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LATERALLY LOADED PLANE STRUCTURES AND STRUCTURES CURVED IN SPACE

By Frank Baron and James P. Michalos, Assoc. Members, ASCE

STRUCTURAL DIVISION

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PAPERS

LATERALLY LOADED PLANE STRUCTURES AND STRUCTURES CURVED IN SPACE

By Frank Baron[1] and James P. Michalos,[2]
Assoc. Members, ASCE

Synopsis

An analytical procedure is presented for determining the effects of loads normal to the plane of such structures as arches, bents, and balcony girders. This procedure is called the "shear and torsion analogy." In addition, a procedure is presented for determining moments and shears in structures curved or segmental in space and continuous between two supports. These structures may be subjected to loads in any direction and to moments about any axis. The procedure is general, and consists of an extension, into space, of the shear and torsion analogy and of the column analogy.

Introduction

Many curved and segmental structures lie in a plane, are loaded normal to their plane, and are continuous between two supports. Arches, balcony girders, and bents are examples of structures that may be loaded in this way. Analytical studies of structures thus loaded have been conducted in the United States by Milo S. Ketchum,[3] Hon. M. ASCE, I. Oesterblom,[4] M. ASCE, Robert B. B. Moorman,[5,6] M. ASCE, and Mervin B. Hogan.[7,8] Similar studies have been

Note.—Written comments are invited for publication; the last discussion should be submitted by July 1, 1951.

[1] Prof., Civ. Eng., Technological Inst., Northwestern Univ., Evanston, Ill.

[2] Associate Prof., Dept. of Civ. Eng., Iowa State College, Ames, Iowa.

[3] "The Design of Walls, Bins, and Grain Elevators," by Milo S. Ketchum, McGraw-Hill Book Co., Inc., New York, N. Y., 3d Ed., 1929.

[4] "Bending and Torsion in Horizontally Curved Beams," by I. Oesterblom, *Proceedings*, A. C. I., Vol. 28, 1932.

[5] "A Semi-Graphical Method of Analysis for Horizontally Curved Beams," by Robert B. B. Moorman, *Bulletin No. 29*, Eng. Experiment Station, Univ. of Missouri, Columbia, Mo., 1938.

[6] "Influence Lines for Horizontally Curved Fixed-End Beams of Circular-Arc Plan," by Robert B. B. Moorman, *Bulletin No. 35*, Eng. Experiment Station, Univ. of Missouri, Columbia, Mo., 1947.

[7] "Circular Beams Loaded Normal to the Plane of Curvature," by Mervin B. Hogan, *Transactions*, ASME, Vol. 60, 1938, p. A-81.

[8] "Circular Beams Loaded Normal to the Plane of Curvature—2," by Mervin B. Hogan, *ibid.*, Vol. 66, 1944, p. A-51.

conducted abroad by A. H. Gibson and E. G. Ritchie;[9] A. J. S. Pippard, M. ASCE, and F. L. Barrow;[10,11] Georg Unold;[12] and others.

An analytical procedure is presented for determining moments and shears in a plane structure continuous between two supports. The structure considered is subjected to loads normal to its plane and moments about axes in its plane. It may have any shape in plan and any variation in cross section along its length. The procedure is called the shear and torsion analogy. The expressions are somewhat similar in form to those of the column analogy,[13] the results in both cases being interpreted in terms of a pressure line concept.

Some structures are continuous between two supports but do not lie in a plane. These structures may be curved or segmental in space and may be subjected to loads or deformations in any direction. Many odd-shaped structures frequently used in urban freeway construction are examples of such structures. Pipe lines are additional examples. Some analytical studies of pipe lines have been conducted by William Hovgaard,[14,15] A. M. Wahl,[16] S. W. Spielvogel,[17] M. ASCE, and S. Kameros.[18] This paper presents an analytical procedure for determining moments and shears in a structure curved or segmental in space and continuous between two supports. The structure may be subjected to loads in any direction and moments about any axis. In addition, it may have any shape in space and any variation in cross section along its length. The procedure is general, and for structures lying in a plane it reduces to the shear and torsion analogy or the column analogy.

The Shear and Torsion Analogy

In the analysis of statically indeterminate structures, the following requirements must be satisfied:

a. Statics.—For any structure or any part of a structure the sum of the forces in any direction must equal zero, and the sum of the forces times their respective distances to any axis must equal zero.

b. Geometry.—Angle changes, slopes, and displacements must be consistent with the requirements of continuity. For any closed circuit the sum of the angle changes about any axis must equal zero. In addition, if the angle

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[9] "A Study of the Circular-Arc Bow-Girder," by A. H. Gibson and E. G. Ritchie, Constable and Co., London, 1914.

[10] "The Analysis of Engineering Structures," by A. J. S. Pippard and J. F. Baker, Longmans, Green and Co., New York, N. Y., 1936.

[11] "The Stress Analysis of Bow Girders," by A. J. S. Pippard and F. L. Barrow, H. M. Stationery Office, London, 1926.

[12] "Der Kreistrager," by Georg Unold, *Forschungsarbeiten No. 255*, Verein Deutscher Ingenieure, Berlin, 1922.

[13] "The Column Analogy," by Hardy Cross, *Bulletin No. 215*, Eng. Experiment Station, Univ. of Illinois, Urbana, Ill., 1930.

[14] "Stresses in Three-Dimensional Pipe Bends," by William Hovgaard, *Transactions*, ASME, Vol. 57, 1935, FSP-57-12, p. 401.

[15] "Further Studies of Three-Dimensional Pipe Bends," by William Hovgaard, *ibid.*, Vol. 59, 1937, FSP-59-13, p. 647.

[16] "Stresses and Reactions in Expansion Pipe Bends," by A. M. Wahl, *ibid.*, Vol. 49–50, 1928, FSP-50-49, p. 241.

[17] "Piping Stress Calculations Simplified," by S. W. Spielvogel, McGraw-Hill Book Co., Inc., New York, N. Y., 1943.

[18] "Application of the Elastic-Point Theory to Piping Stress Calculations," by S. W. Spielvogel and S. Kameros, *Transactions*, ASME, Vol. 57, 1935, FSP-57-10, p. 165.

changes are small, the sum of the angle changes times their respective distances to any axis must equal zero.

Geometry must be related to statics through the known or assumed properties of materials. No uncertainties are involved in the requirements of statics and geometry. Some uncertainties, however, may exist in the relationship between statics and geometry due to the characteristics of a material. This paper does not deal with uncertainties in the properties of materials. It is assumed that the properties necessary for computing the relationships between statics and geometry are known. Furthermore, the effects of shear deformations and of changes in length of an element of a structure are considered negligible. If desired, adjustments in the succeeding analyses may be made for these effects.

In general, two classes of loading may be considered for curved structures lying in a plane and continuous between two supports. These loadings are shown in Fig. 1(*a*) and are as follows:

(A) Loads applied in the plane of the structure and moments about axes normal to this plane. This class of loading is considered in the column analogy.

(B) Loads applied normal to the plane of the structure and moments about axes in this plane. This class of loading is particularly considered in the present section of the paper.

For each class of loading, the forces and moments acting on a cross section of the structure are shown in Fig. 1(*b*) and are represented as vectors. The angle changes, rotations, and displacements at a section of the structure are shown in Fig. 1(*c*) and are also represented as vectors. The sign convention used in this paper is illustrated in the same figure. Forces and moments are positive on a front face when they are in the direction of an axis. They are positive on a back face when they are against the direction of an axis. The same sign convention is used for rotations and displacements. The angle changes between adjacent cross sections of an element are positive when they are in the direction of an axis. The statical and geometrical relationships are referred to rectangular axes.

Structures loaded as in class (A) or as in class (B) may be analyzed by a procedure consisting of two parts as follows:

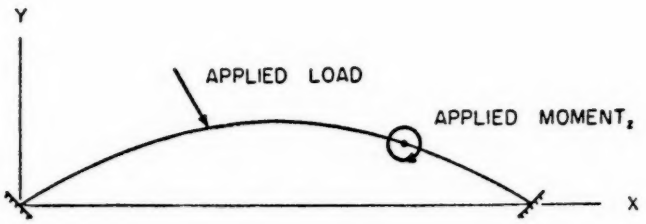
1. Assume any statically possible distribution of moments along the structure. Check to see if errors exist in the geometry of the deflected structure.

2. If such errors exist, add (or subtract) a distribution of moments that corrects these errors. This distribution, however, must not introduce errors in statics.

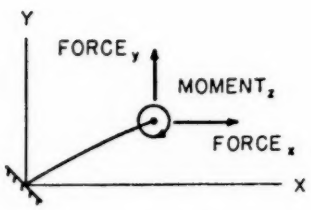
Such a procedure automatically satisfies the requirements of statics. The following discussion is restricted to the analysis of laterally loaded plane structures. The final moments and shear at any section of the structure are obtained as follows:

LOADS AS IN (A)

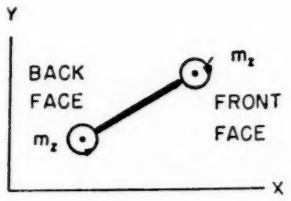
THIS CLASS OF LOADING IS CONSIDERED IN THE COLUMN ANALOGY.



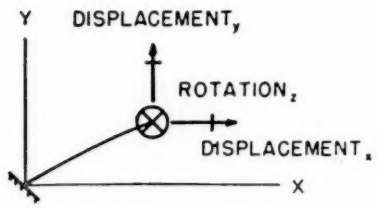
(a) LOADS IN THE PLANE OF THE STRUCTURE AND MOMENTS ABOUT AXES NORMAL TO THIS PLANE.



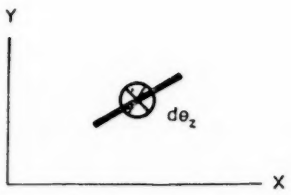
(b) FORCES AND A MOMENT ACTING ON A CROSS SECTION OF THE STRUCTURE



MOMENTS, m_z, ON ADJACENT CROSS SECTIONS OF AN ELEMENT



(c) DISPLACEMENTS AND A ROTATION AT A CROSS SECTION OF THE STRUCTURE

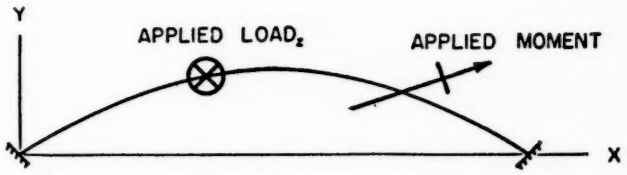


ANGLE CHANGE, $d\theta_z$, ABOUT AN AXIS NORMAL TO THE PLANE OF THE STRUCTURE

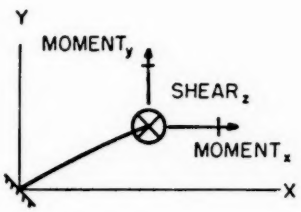
FIG. 1.—STATICAL AND GEOMETRICAL RELATIONSHIPS

LOADS AS IN (B)

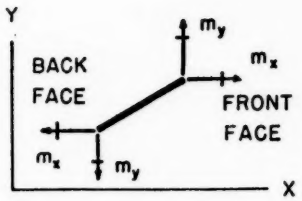
THIS CLASS OF LOADING IS PARTICULARLY CONSIDERED IN THE FIRST SECTION OF THE PAPER.



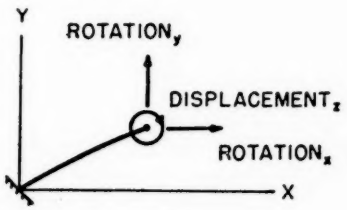
(a) LOADS NORMAL TO THE PLANE OF THE STRUCTURE AND MOMENTS ABOUT AXES IN THIS PLANE.



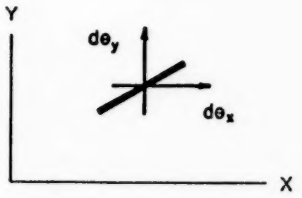
(b) MOMENTS AND A FORCE ACTING ON A CROSS SECTION OF THE STRUCTURE.



MOMENTS, m_x AND m_y, ON ADJACENT CROSS SECTIONS OF AN ELEMENT.



(c) ROTATIONS AND A DISPLACEMENT AT A CROSS SECTION OF THE STRUCTURE.



ANGLE CHANGES, $d\theta_x$ AND $d\theta_y$, ABOUT AXES IN THE PLANE OF THE STRUCTURE.

Moment—

$$m_x = m_{xo} - m_{xi} \quad \ldots\ldots (1a)$$

moment—

$$m_y = m_{yo} - m_{yi} \quad \ldots\ldots (1b)$$

and shear—

$$v_z = v_{zo} - v_{zi} \quad \ldots\ldots (1c)$$

The subscripts o and i are associated with the assumed distributions and the correction distributions, respectively.

Now consider the distributions of correction moments and shears. Fig. 2(a) shows the correction shears and moments at a support and at a section of the structure. The relationships between these forces and moments are summarized in the figure. The same structure is shown in Fig. 2(b) but an imaginary arm is attached to an end of the structure. This arm is introduced for convenience only. It is considered rigid and extends to a position that will be referred to as a centroid. The position of this centroid and the properties of the structure with respect to this centroid are defined subsequently. Fig. 2(b) also shows the correction moments and shear applied at the centroid. By statics the correction moments and shear at a section of the structure are as follows:

Moment—

$$m_{xi} = m_{xic} + v_{zic}\, y_c \quad \ldots\ldots (2a)$$

moment—

$$m_{yi} = m_{yic} - v_{zic}\, x_c \quad \ldots\ldots (2b)$$

and shear—

$$v_{zi} = v_{zic} \quad \ldots\ldots (2c)$$

The subscript c is associated with the correction moments and shear at the centroid and with distances measured from the axes at the centroid. An inspection of Eqs. 2a, 2b, and 2c shows that the distributions of correction moments and shears are planar.

The requirements of geometry are now restated. The sum of the angle changes about any axis in the plane of the structure must equal zero; thus:

$$\sum d\theta_x = \sum d\theta_{xo} - \sum d\theta_{xi} = 0 \quad \ldots\ldots (3a)$$

and

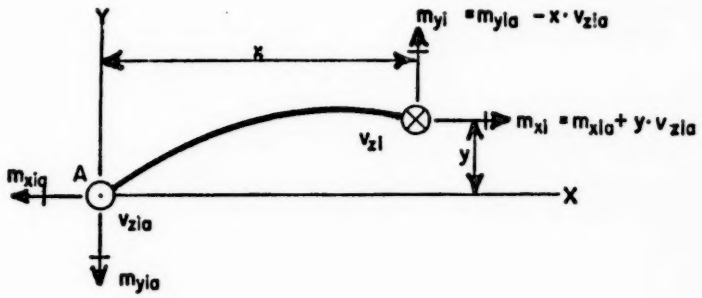
$$\sum d\theta_y = \sum d\theta_{yo} - \sum d\theta_{yi} = 0 \quad \ldots\ldots (3b)$$

The sum of the angle changes, times their respective distances to any axis normal to the plane, must equal zero:

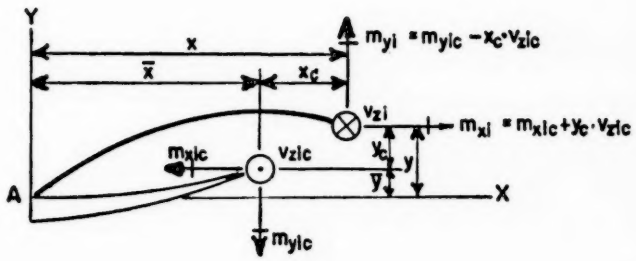
$$(\sum y\, d\theta_x - \sum x\, d\theta_y) = (\sum y\, d\theta_{xo} - \sum x\, d\theta_{yo}) - (\sum y\, d\theta_{xi} - \sum x\, d\theta_{yi}) = 0 \quad \ldots (3c)$$

Thus, the angle changes as a result of the correction moments must balance those caused by the assumed moments.

Fig. 3(*a*) shows the angle changes associated with assumed or correction distributions of moments. These angle changes are represented as vectors. In this case each cross section has a principal axis in the plane of the structure.



(a) CORRECTION MOMENTS AND SHEARS IN THE STRUCTURE DUE TO MOMENTS AND SHEAR AT END A.



$$y = \bar{y} + y_c \qquad m_{xia} = m_{xic} - \bar{y} \cdot v_{zic}$$

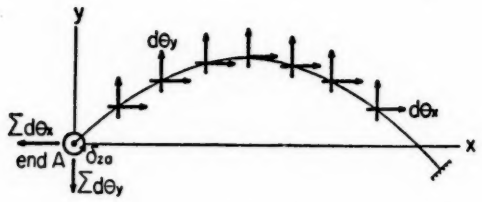
$$x = \bar{x} + x_c \qquad m_{yia} = m_{yic} + \bar{x} \cdot v_{zic}$$

$$v_{zia} = v_{zic} = v_{zi}$$

(b) CORRECTION MOMENTS AND SHEARS IN THE STRUCTURE DUE TO MOMENTS AND SHEAR AT THE CENTROID

FIG. 2.—CORRECTION MOMENTS AND SHEARS IN A LATERALLY LOADED PLANE STRUCTURE

The general case of an element in space and oblique to the three orthogonal axes is considered subsequently. The rotations and displacement at an end A of the structure due to the assumed moments and the correction moments



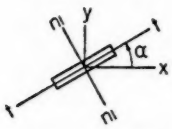
$$\Sigma d\theta_x = \Sigma\left(m_x \frac{ds}{K_{xx}} + m_y \frac{ds}{K_{xy}}\right) = \Sigma(m_x\, da_{xx} + m_y\, da_{xy})$$

$$\Sigma d\theta_y = \Sigma\left(m_y \frac{ds}{K_{yy}} + m_x \frac{ds}{K_{yx}}\right) = \Sigma(m_y\, da_{yy} + m_x\, da_{yx})$$

$$\delta_{za} = \Sigma y\left(m_x \frac{ds}{K_{xx}} + m_y \frac{ds}{K_{xy}}\right) - \Sigma x\left(m_y \frac{ds}{K_{yy}} + m_x \frac{ds}{K_{yx}}\right)$$

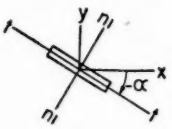
$$= \Sigma y(m_x\, da_{xx} + m_y\, da_{xy}) - \Sigma x(m_y\, da_{yy} + m_x\, da_{yx})$$

(a) Rotations and Displacement at End A



$K_n = EI_t$

$K_t = GJ$



Positive Slope of Segment Negative Slope of Segment

$$K_{xx} = \frac{K_n K_t}{K_n \cos^2\alpha + K_t \sin^2\alpha} \qquad K_{yy} = \frac{K_n K_t}{K_n \sin^2\alpha + K_t \cos^2\alpha}$$

$$K_{xy} = \frac{K_n K_t}{(K_n - K_t)\sin\alpha\cos\alpha}$$

(b) Summary of Stiffness Factors

FIG. 3.—GEOMETRICAL RELATIONSHIPS FOR LATERALLY LOADED PLANE STRUCTURES

are summarized in Fig. 3(*a*). Since the angle changes due to the correction moments must balance those due to the assumed moments, the following equations of geometry may be written:

For rotations about the *x*-axis—

$$m_{zia}\Sigma da_{xx} + m_{yia}\Sigma da_{xy} + v_{zia}\Sigma(y\, da_{xx} - x\, da_{xy}) = \Sigma(m_{zo}\, da_{xx} + m_{yo}\, da_{xy}) \ldots (4a)$$

for rotations about the *y*-axis—

$$m_{zia}\Sigma da_{yx} + m_{yia}\Sigma da_{yy} + v_{zia}\Sigma(y\, da_{yx} - x\, da_{yy}) = \Sigma(m_{yo}\, da_{yy} + m_{zo}\, da_{yx}) \ldots (4b)$$

and, for displacements along the *z*-axis,

$$m_{zia}\Sigma(y\, da_{xx} - x\, da_{yx}) + m_{yia}\Sigma(y\, da_{xy} - x\, da_{yy}) + v_{zia}[\Sigma y(y\, da_{xx} - x\, da_{xy}) - \Sigma x(y\, da_{yx} - x\, da_{yy})] = \Sigma y(m_{zo}\, da_{xx} + m_{yo}\, da_{xy}) - \Sigma x(m_{yo}\, da_{yy} + m_{zo}\, da_{yx}) \ldots (4c)$$

Moments about one axis contribute to the angle changes about both axes. In Eqs. 4*a*, 4*b*, and 4*c*,

$$\left.\begin{aligned} da_{xx} &= \frac{ds}{K_{xx}} \\ da_{yy} &= \frac{ds}{K_{yy}} \\ da_{xy} &= \frac{ds}{K_{xy}} \\ da_{yx} &= \frac{ds}{K_{yx}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The terms da_{xx} and da_{xy} represent angle changes about an x-axis due to unit moments about the x-axis and the y-axis, respectively. Similarly, da_{yy} and da_{yx} represent angle changes about a y-axis due to unit moments about the y-axis and x-axis, respectively. The unit moments are at the ends of an element which is ds in length and is oblique to the rectangular axes. For an element one unit in length, K_{xx} and K_{xy} represent moments about the x-axis and the y-axis, respectively. Each moment produces a unit angle change about the x-axis. In the same way, K_{yy} and K_{yx} represent moments about the y-axis and the x-axis, respectively. These moments produce unit angle changes about the y-axis. It may be shown that $K_{yx} = K_{xy}$ and $da_{yx} = da_{xy}$. Thus, K_{xx}, K_{yy}, and K_{xy} are measures of the stiffness of the element and for convenience are called stiffness factors. Algebraic expressions for these factors are given in Fig. 3(*b*) in terms of the normal and tangential stiffnesses of the element.

Now inspect Eqs. 4*a*, 4*b*, and 4*c* and observe that the unknowns in these equations are the correction moments and shear at end A of the structure. From the relationships shown in Fig. 2, these equations may be written in terms of the correction moments and shear at the centroid. If the distances $\bar{x}$ and $\bar{y}$ to the centroidal axes are now selected such that—

$$\sum (y_c\, da_{xx} - x_c\, da_{xy}) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6a)$$

and

$$\sum (y_c\, da_{yx} - x_c\, da_{yy}) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6b)$$

—then, Eqs. 4*a*, 4*b*, and 4*c* may be written as follows:

$$m_{xic} \sum da_{xx} + m_{yic} \sum da_{xy} = \sum (m_{xo}\, da_{xx} + m_{yo}\, da_{xy}) \ldots\ldots\ldots (7a)$$

$$m_{xic} \sum da_{yx} + m_{yic} \sum da_{yy} = \sum (m_{yo}\, da_{yy} + m_{xo}\, da_{yx}) \ldots\ldots\ldots (7b)$$

and

$$v_{zic} \left[\sum y_c\, (y_c\, da_{xx} - x_c\, da_{xy}) - \sum x_c\, (y_c\, da_{yx} - x_c\, da_{yy})\right] = \sum y_c\, (m_{xo}\, da_{xx} + m_{yo}\, da_{xy}) - \sum x_c\, (m_{yo}\, da_{yy} + m_{xo}\, da_{yx}) \ldots (7c)$$

Thus, the position of the centroid is so defined that if a shear, v_{zic}, is applied at this centroid it does not contribute to rotations at end A. If the correction moments and shear at the centroid are known, the correction moments and shear at any section of the structure may be obtained by statics. In the follow-

ing, the form of the expressions for these corrections is obtained by the concept that permits the interchange of computations of geometry with those of statics.

This concept is based on the definitions of a small angle and of a moment. By definition of a small angle, a displacement is equal to the product of an angle change and the normal distance to the line along which the displacement occurs. A moment about an axis is defined as the product of a force and the normal distance to the axis. Thus, computations of geometry involving small angles are interchangeable with those of statics. This is formally stated as follows:

(*a*) An angle change or rotation about an axis may be treated as a force along the axis of rotation.

(*b*) A displacement along an axis may be treated as a moment about the axis of displacement. (This generalized statement was made by Hardy Cross, Hon. M. ASCE, at Yale University, New Haven, Conn.)

Consequently, in computing the geometry of a deflected structure, the rotations and displacements at a cross section may be treated as analogous shears and moments. This treatment may readily be visualized by reinspecting Fig. 1.

From the foregoing interchange, the requirements of geometry for a closed circuit lying in a plane may be restated as analogous requirements of statics. The analogous requirements are stated algebraically:

$$\sum dp_x = \sum dp_{xo} - \sum dp_{xi} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (8a)$$

$$\sum dp_y = \sum dp_{yo} - \sum dp_{yi} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (8b)$$

and

$$(\sum y\, dp_x - \sum x\, dp_y) = (\sum y\, dp_{xo} - \sum x\, dp_{yo}) - (\sum y\, dp_{xi} - \sum x\, dp_{yi}) = 0 \ldots (8c)$$

Eqs. 8*a*, 8*b*, and 8*c* are equations of statics and are analogous to the previous equations of geometry (Eqs. 3*a*, 3*b*, and 3*c*). Note that the analogous forces associated with the correction moments must balance those associated with the assumed moments.

It is now convenient to consider an imaginary strut that has the same shape in plan as the real structure. This strut is very short and is subjected to shear forces lying in the top plane of the strut. The forces are analogous to the corresponding angle changes associated with the assumed distributions of moments. The angle changes $m_{xo}\, da_{xx}$ and $m_{yo}\, da_{xy}$ are the analogous forces dp_{xo} in the x-direction. The angle changes $m_{yo}\, da_{yy}$ and $m_{xo}\, da_{yx}$ are the analogous forces dp_{yo} in the y-direction. Since a shear force is equal to a stress times a corresponding differential area, the assumed moments may be interpreted as analogous shear stresses in the top plane of the strut and the terms da_{xx}, da_{yy}, and da_{xy} may be interpreted as the corresponding differential areas. Consequently, the imaginary strut may be considered to have directional properties. The forces in the top plane of the strut are balanced by forces on cross sections parallel to this plane. Therefore, the forces on such cross sections are analogous to the angle changes associated with the correction moments. The forces dp_{xi} are analogous to the angle changes $m_{xi}\, da_{xx}$ and $m_{yi}\, da_{xy}$. The forces dp_{yi}

are analogous to the angle changes $m_{yi}\,da_{yy}$ and $m_{xi}\,da_{yx}$. Thus, the shear stresses on cross sections parallel to the top plane of the strut may be interpreted as correction moments in the real structure.

It was previously shown that the distributions of correction moments are planar; but if an imaginary strut is loaded as in the preceding case and if the shear stresses across a section of the strut are computed by the following expressions—

$$s_x = \frac{P'_x}{A'_x} + \frac{M_z}{J_z}\,y_c \qquad (9a)$$

and

$$s_y = \frac{P'_y}{A'_y} - \frac{M_z}{J_z}\,x_c \qquad (9b)$$

—the distributions of these stresses are also planar. In each expression, the first term represents an average stress due to a direct shear and the second term represents a stress due to a torque about an axis normal to the top plane of the strut, J_z being the torsional moment of inertia about the z-axis.

As a result of the preceding discussion, it is now possible to state that the correction moments in the real structure can be computed in the same way as shear stresses in an analogous strut:

$$m_{xi} = \frac{P'_x}{A'_x} + \frac{M_z}{J_z}\,y_c \qquad (10a)$$

and

$$m_{yi} = \frac{P'_y}{A'_y} - \frac{M_z}{J_z}\,x_c \qquad (10b)$$

It could also be shown that the correction shears in the real structure can be computed by means of the expression:

$$v_{zi} = \frac{M_z}{J_z} \qquad (10c)$$

The terms of Eqs. 10*a*, 10*b*, and 10*c* must be so defined that the correction moments adjust for the errors in geometry due to the assumed distributions of moments. These terms were obtained by means of the equations of geometry, Eqs. 7*a*, 7*b*, and 7*c*, and the relationships of statics shown in Fig. 2.

All terms involved in the analytical procedure are summarized in Fig. 4(*a*). Inspect Fig. 4(*a*) and interpret the expressions in the summary. For convenience first consider a structure that has an axis of symmetry parallel to the y-axis. For such a structure it may be shown that A_{xy} is equal to zero and that $P'_x = P_x$ and $P'_y = P_y$. The terms P_x and P_y represent the relative rotations of one end of the structure with respect to the other. These rotations are due to the assumed distributions of moments and are treated as total shear forces in the top plane of the strut. For an unsymmetrical structure corrections must be subtracted from P_x and P_y to obtain P'_x and P'_y, respectively.

Now inspect the term M_z. In the real structure, M_z represents a displacement along an axis normal to the plane of the structure. This displacement occurs at the defined centroid and is associated with the angle changes due to the

$$v_z = v_{zo} - v_{zi} \qquad v_{zi} = \frac{M_z}{J_z}$$

$$m_x = m_{xo} - m_{xi} \qquad m_{xi} = \frac{P'_x}{A'_x} + \frac{M_z}{J_z} y_c$$

$$m_y = m_{yo} - m_{yi} \qquad m_{yi} = \frac{P'_y}{A'_y} - \frac{M_z}{J_z} x_c$$

$$A_x = \Sigma da_{xx}$$

$$A_y = \Sigma da_{yy}$$

$$A_{xy} = \Sigma da_{xy}$$

$$A_{xy} = A_{yx}$$

$$J_z = I_x + I_y - 2I_{xy}$$

$$I_x = \Sigma y^2 da_{xx} - \bar{y}^2 A_x \qquad A'_x = A_x - \frac{A_{xy}A_{xy}}{A_y}$$

$$I_y = \Sigma x^2 da_{yy} - \bar{x}^2 A_y \qquad A'_y = A_y - \frac{A_{xy}A_{xy}}{A_x}$$

$$I_{xy} = \Sigma xy\, da_{xy} - \bar{x}\bar{y} A_{xy}$$

$$K_n = EI$$

$$da_{xx} = \frac{ds}{K_{xx}} \qquad K_{xx} = \frac{K_n K_t}{K_n \cos^2\alpha + K_t \sin^2\alpha} \qquad K_t = GJ$$

$$da_{yy} = \frac{ds}{K_{yy}} \qquad K_{yy} = \frac{K_n K_t}{K_n \sin^2\alpha + K_t \cos^2\alpha}$$

$$da_{xy} = \frac{ds}{K_{xy}} \qquad K_{xy} = \frac{K_n K_t}{(K_n - K_t)\sin\alpha\cos\alpha}$$

$$P'_x = P_x - \frac{A_{xy}}{A_y} P_y \qquad P_x = \Sigma m_{xo} da_{xx} + \Sigma m_{yo} da_{xy} \qquad M_z = M_{zx} - M_{zy}$$

$$P'_y = P_y - \frac{A_{xy}}{A_x} P_x \qquad P_y = \Sigma m_{yo} da_{yy} + \Sigma m_{xo} da_{xy}$$

$$M_{zx} = \Sigma y(m_{xo} da_{xx} + m_{yo} da_{xy}) - \bar{y} P_x$$

$$M_{zy} = \Sigma x(m_{yo} da_{yy} + m_{xo} da_{xy}) - \bar{x} P_y$$

$$x_c = x - \bar{x} \qquad \bar{x} = -\frac{C'_y}{A'_y} \qquad C'_y = C_y - \frac{A_{xy}}{A_x} C_x \qquad C_y = \Sigma y\, da_{xy} - \Sigma x\, da_{yy}$$

$$y_c = y - \bar{y} \qquad \bar{y} = \frac{C'_x}{A'_x} \qquad C'_x = C_x - \frac{A_{xy}}{A_y} C_y \qquad C_x = \Sigma y\, da_{xx} - \Sigma x\, da_{xy}$$

(a) GENERAL RELATIONSHIPS

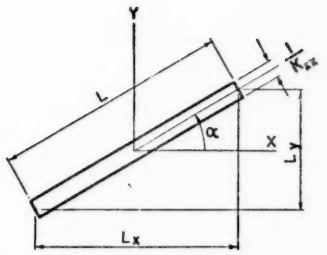
FIG. 4.—SUMMARY FOR STRUCTURES LYING IN A

ELASTIC AREA = $\frac{L}{K_{xx}} = d\alpha_{xx}$

$i_x = \frac{1}{12} d\alpha_{xx} L_y^2$

$i_y = 0$

$i_{xy} = 0$

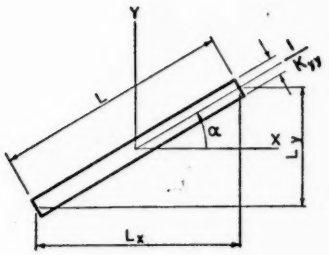


ELASTIC AREA = $\frac{L}{K_{yy}} = d\alpha_{yy}$

$i_x = 0$

$i_y = \frac{1}{12} d\alpha_{yy} L_x^2$

$i_{xy} = 0$

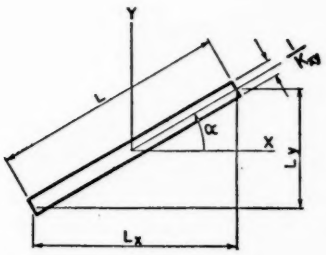


ELASTIC AREA = $\frac{L}{K_{xy}} = d\alpha_{xy}$

$i_x = 0$

$i_y = 0$

$i_{xy} = \frac{1}{12} d\alpha_{xy} L_x L_y$



NOTE: K_{xy} IS POSITIVE IF THE SEGMENT SLOPES ↗, AND NEGATIVE IF IT SLOPES ↖ THEREFORE, I_{xy} FOR A SYMMETRICAL STRUCTURE IS NOT EQUAL TO ZERO.

(b) CONVENIENT RELATIONSHIPS OF FINITE ELASTIC AREAS

PLANE AND LOADED NORMAL TO THEIR PLANE

assumed distributions of moments. In the analogous strut, M_z represents a moment of the analogous forces lying in the top plane of the strut. This moment is about an axis normal to the plane of the strut and located at the centroid of the strut. This moment may be obtained by first computing the moment about an arbitrary axis and then subtracting a correction as indicated in Fig. 4(*a*).

Member	Length	$\frac{l}{K_{yy}}$	$\frac{l}{K_{xx}}$	$\frac{l}{K_{xy}}$	da_{yy}	da_{xx}	da_{xy}	x	y	yda_{xx}	xda_{xy}	yda_{xy}	xda_{yy}
										✓	✓✓	✓	✓✓
Totals					A_y	A_x	A_{xy}			✓ − ✓✓ = C_x		✓ − ✓✓ = C_y	
Correct for Dissymmetry (subtract)					$\frac{A_{xy}}{A_x} A_{xy}$	$\frac{A_{xy}}{A_y} A_{xy}$				$\frac{A_{xy}}{A_y} C_y$		$\frac{A_{xy}}{A_x} C_x$	
					A'_y	A'_x				C'_x		C'_y	

Centroid: $\bar{x} = -\frac{C'_y}{A'_y}$; $\bar{y} = \frac{C'_x}{A'_x}$

Corrections to Centroid (Subtract)

Corrected

$$v_{zi} = \frac{M_z}{J_z} \qquad v_z = v_{zo} - v_{zi}$$

$$m_{xi} = \frac{P'_x}{A'_x} + \frac{M_z}{J_z} y_c \qquad m_x = m_{xo} - m_{xi}$$

$$m_{yi} = \frac{P'_y}{A'_y} - \frac{M_z}{J_z} x_c \qquad m_y = m_{yo} - m_{yi}$$

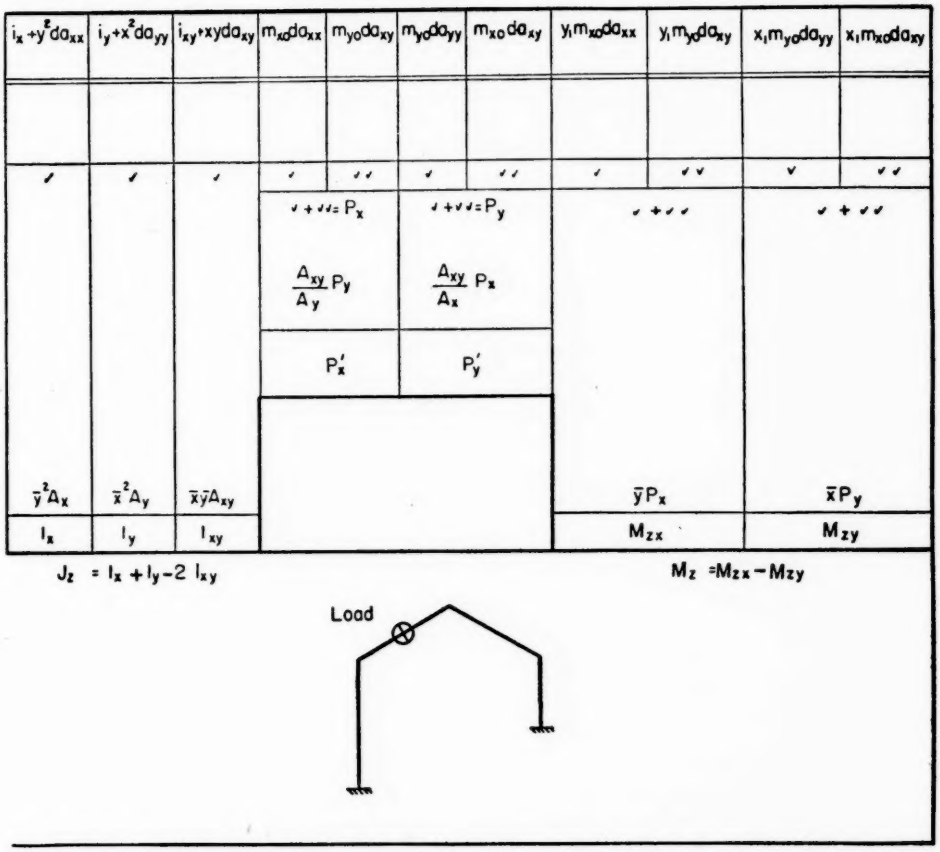
FIG. 5.—TABULAR FORM FOR ANALYSIS OF

Next, inspect the terms A'_x and A'_y. Again, for convenience, consider the case of a symmetrical structure. For such a structure A_{xy} is equal to zero; therefore, $A'_x = A_x$ and $A'_y = A_y$. The terms A_x and A_y represent the relative rotations of one end of the structure with respect to the other due to moments of one unit about an x-axis and a y-axis, respectively. In the analogous strut, A_x and A_y represent total areas. If desired, A_x and A_y may be interpreted as measures of stiffness to shear deformations along the x-axis and the y-axis, respectively. For an unsymmetrical structure corrections must be subtracted from A_x and A_y to obtain A'_x and A'_y, respectively.

Finally, consider the term J_z. This term is written as follows:

$$J_z = I_x + I_y - 2\,I_{xy} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

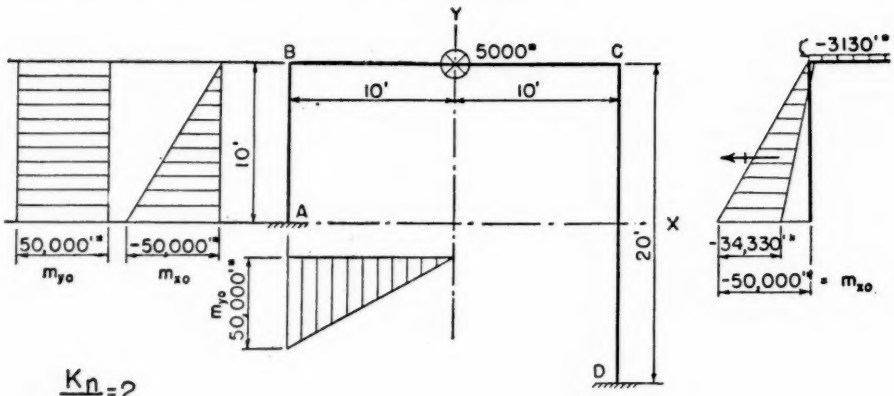
and represents a displacement along a z-axis at the centroid. This displacement is due to a unit shear along the same axis. Since a unit shear at the centroid does not contribute to rotation of one end of the structure with respect to the



PLANE STRUCTURES SUBJECTED TO LATERAL LOADS

other, the displacement at end A is equal to that at the centroid. In Eq. 11 $I_x - I_{xy}$ represents a contribution to the total displacement at end A and is due to moments about x-axes resulting from the unit shear at the centroid. The contribution due to moments about y-axes is represented by $I_y - I_{xy}$. In the analogous strut, the expressions for I_z, I_y, and I_{xy} resemble the familiar transfer equations for moments and products of inertia. The term J_z is similar to the expression for a polar moment of inertia except that I_{xy} is included. If desired, J_z may be interpreted as a measure of the stiffness of the strut to a torsional deformation about a z-axis at the centroid.

THE MOMENT CURVES BELOW ARE CONSISTENT WITH ASSUMING THE STRUCTURE CANTILEVERED FROM A



$\frac{K_n}{K_t}=2$

LOADED STRUCTURE.

Member	Length	$\frac{K_n}{K_{yy}}$	$\frac{K_n}{K_{xx}}$	da_{yy}	da_{xx}	x	y
A B	10.0	2.0	1.0	20.0	10.0	-10	5
B C	20.0	1.0	2.0	20.0	40.0	0	10
C D	20.0	2.0	1.0	40.0	20.0	10	0
Totals				$A_y=80.0$	$A_x=70.0$		

Centroid: $\bar{x}=\frac{C_y}{A_y}=\frac{200}{80}=2.5$ ft. ; $\bar{y}=\frac{C_x}{A_x}=\frac{450}{70}=6.43$ ft.

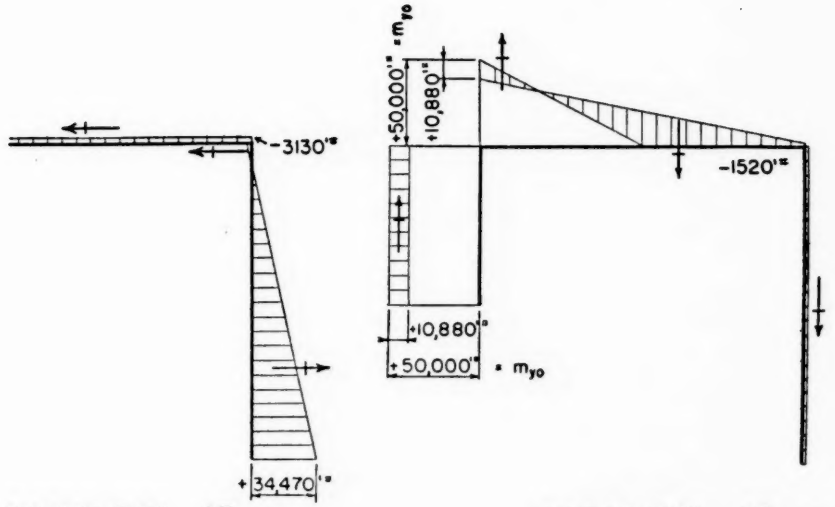
$$v_{zi}=\frac{M_z}{J_z}=\frac{1556.0\times10^4}{8273}=1880 \text{ lb.}$$

At A:

$$m_{xi}=\frac{P_x}{A_x}+\frac{M_z}{J_z}y_c=\frac{-25\times10^4}{70}+1880(-6.43)=-3750-12{,}100=-15{,}670 \text{ ft-lb.}$$

$$m_{yi}=\frac{P_y}{A_y}-\frac{M_z}{J_z}x_c=\frac{125.0\times10^4}{80}-1880(-12.50)=15{,}620+23{,}500=39{,}120 \text{ ft-lb.}$$

FIG. 6.—UNSYMMETRICAL BENT



DISTRIBUTION OF m_x DISTRIBUTION OF m_y

xda_{yy}	yda_{xx}	$i_x+y^2da_{xx}$	$i_y+x^2da_{yy}$	$m_{xo}da_{xx}$ x10⁴	$m_{yo}da_{yy}$ x10⁴	$y_i m_{xo}da_{xx}$ x10⁴	$x_i m_{yo}da_{yy}$ x10⁴
-200	50	333	2000	-25.0	100.0	-83.2	-1000
0	400	4000	667	0	25.0	0	-166
400	0	667	4000	0	0	0	0
C_y=-200	C_x=450	5000	6667	P_x=-25.0	P_y=125.0	-83.2	-1166
Corrections to Centroid		$\bar{y}^2A_x$=2894	$\bar{x}^2A_y$=500			$\bar{y}P_x$=-161	$\bar{x}P_y$=312
Corrected		I_x=2106	I_y=6167			M_{zx}=77.8	M_{zy}=-1478

$I_x + I_y = J_z = 8273$ $M_{zx} - M_{zy} = M_z = 1556.0$

THE MOMENT CURVES ABOVE ARE OBTAINED BY CORRECTING THE DISTRIBUTIONS OF MOMENTS CONSISTENT WITH ASSUMING THE STRUCTURE CANTILEVERED FROM A.

$$v_z = v_{z\bar{o}} - v_{zi} = 5000 - 1880 = 3120 \text{ lb.}$$

$$m_x = m_{xo} - m_{xi} = -50,000 - (-15,670) = -34,330 \text{ ft-lb.}$$

$$m_y = m_{yo} - m_{yi} = 50,000 - 39,120 = 10,880 \text{ ft-lb.}$$

For aid in making computations, a tabular form is presented in Fig. 5, which indicates a convenient sequence for computing the properties of the structure and the analogous forces. The final shear and moments at any section of the structure may be computed as shown in the lower part of the figure. The values of x, y and x_1, y_1 may be taken from any convenient pair of axes. If the structure is symmetrical to one of these axes, it is necessary to compute the properties of only one half the structure and to double the sums. In the analytical procedure x and y were defined as distances to differential elements. If segments of finite length are considered, x and y represent distances to the centroids of the elastic areas da_{xx}, da_{yy}, and da_{xy}. For a segment of constant cross section, the centroids of the elastic areas coincide with the centroid of the segment. However, in computing I_x, I_y, and I_{xy}, it is necessary to add corrections i_x, i_y, and i_{xy} to the values of $y^2\,da_{xx}$, $x^2\,da_{yy}$, and $x\,y\,da_{xy}$, respectively. These corrections are moments and products of inertia of the elastic areas about their own centroidal axes. The resulting expressions such as $i_x + y^2\,da_{xx}$ are analogous to the familiar transfer equations for moments and products of inertia. Fig. 4(*b*) summarizes convenient relationships for computing values of i_x, i_y, and i_{xy} of a segment oblique to the x-axis and the y-axis.

It may be shown that for a finite segment x_1 and y_1 are distances to the centroids of the corresponding angle changes along the segment and not distances to the centroids of the elastic areas.

Numerical Examples.—Fig. 6 shows a rectangular bent of constant cross section with a lateral load of 5,000 lb. The ratio of bending to torsional stiffness is equal to two. For a constant cross section it is not necessary to know the absolute values of K_n and K_t. For a given $\frac{K_n}{K_t}$-ratio, values of K_n/K_{yy} and K_n/K_{xx} may be computed. For convenience such values may be plotted for several ratios of K_n/K_t and for different slopes of a segment. Similarly, curves may be drawn showing values of K_n/K_{xy}. In the computations for the rectangular bent, values of K_n/K_{yy} and K_n/K_{xx} have been used instead of $1/K_{yy}$ and $1/K_{xx}$. As a result all items involving da_{yy} and da_{xx} are in terms of K_n. An inspection of the tabulation shows that in the computations for moments and shear the quantities K_n cancel out. Fig. 6 also shows the final curves of moments. These curves are drawn in two steps: First, the assumed moments, and then planar distributions of correction moments. Note how the correction moments balance the assumed moments. This balance concept lends itself to the development of a sketching technique by which good estimates of moments can be obtained. The procedure may be considered an extension of the pressure line concept as used for an arch with loads in the plane of the arch.[19,20] In addition, the concept can be used in problems involving pure torsion or axial deformations.

Fig. 7 shows an arch with a varying cross section and with a lateral load of one kip at the quarter point. The arch is divided into twenty segments whose horizontal projections measure 5 ft. The analogous moments of inertia and products of inertia about the centroidal axes of the segments are neglected.

[19] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932, p. 289.

[20] *Ibid.*, p. 295.

In addition, the distances to the centroids of the angle changes for each segment are taken as the tabulated distances to the centroid of the segment. The values of K_n/K_t at the various sections, and the relative values of K_n are shown in the table at the bottom of Fig. 7. These values are such as might be probable in a reinforced concrete arch. Since K_n/K_t is not constant, it is now necessary to tabulate values equal to, or proportional to $1/K_{xx}$, $1/K_{yy}$, and $1/K_{xy}$. In this example, proportional values were determined by the relative values of K_n. The final curves of moments are shown in the upper right-hand corner of Fig. 7. Again, the assumed distributions of moments are balanced by planar distributions of correction moments.

A General Procedure for Structures Curved in Space

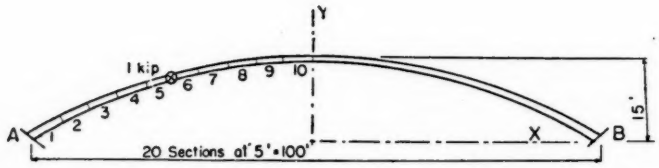
A procedure is presented, in this section, for analyzing a structure that is continuous between two supports but not lying in a plane. This structure may have any shape in space and any variation in cross section along its length. It may be subjected to loads in any direction and to moments about any axis. Three orthogonal axes are chosen for reference, but no part of the structure need lie in the planes defined by these axes. The sign convention used in this section is the same as that used in the preceding section. As before, the effects of shear deformations and changes in length of an element of the structure are considered negligible.

As in the shear and torsion analogy, an analytical procedure consisting of two steps is used. In accordance with this procedure the final moments and shears at any section of the structure are written as follows:

$$\left.\begin{aligned} m_x &= m_{xo} - m_{xi} \\ m_y &= m_{yo} - m_{yi} \\ m_z &= m_{zo} - m_{zi} \end{aligned}\right\} \qquad \left.\begin{aligned} v_x &= v_{xo} - v_{xi} \\ v_y &= v_{yo} - v_{yi} \\ v_z &= v_{zo} - v_{zi} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (12)$$

Consider the distributions of correction moments and shears. Fig. 8(*a*) shows three correction moments and three correction shears applied to the structure. These moments and shears are projected in each plane. The moments are about the x-axis, the y-axis, and the z-axis, and the shears are along axes which, for convenience, are called centroidal axes. Six distances are used to locate the centroidal axes. The positions of these axes and the properties of the structure with respect to these axes are defined subsequently. In each plane the projections of two moments and a shear have been grouped as in the shear and torsion analogy, and the projections of two shears and a moment as in the column analogy. The relationships between these moments and shears, and the correction moments and shears at a section of the structure, are shown in Fig. 8(*a*). Note that the resulting distributions of correction moments and shears are planar.

Fig. 8(*b*) shows rotations about the x-axis, the y-axis, and the z-axis, and displacements along the centroidal axes. Each axis of rotation has been chosen to correspond with that of a moment as shown in Fig. 8(*a*) and each axis of displacement corresponds with that of a shear. The expressions for the rota-



Section	ΔS	$\frac{I}{K_{yy}}$	$\frac{I}{K_{xx}}$	$\frac{I}{K_{xy}}$ (−)	da_{yy}	da_{xx}	da_{xy} (−)	x (−)	y	ydA_{xx}	xdA_{xy}	y^2dA_{xx}	x^2dA_{yy}
1	5.75	0.49	0.22	0.09	2.58	1.28	0.54	47.5	1.46	1.8	25.7	2	5825
2	5.62	0.59	0.32	0.09	3.32	1.80	0.50	42.5	4.16	7.4	21.0	31	6000
3	5.48	0.71	0.48	0.06	3.89	2.64	0.33	37.5	6.56	17.3	12.3	113	5470
4	5.36	0.78	0.58	0.05	4.18	3.09	0.24	32.5	8.66	26.8	7.9	231	4415
5	5.25	0.82	0.64	0.03	4.31	3.34	0.18	27.5	10.46	34.9	4.8	365	3264
6	5.21	0.86	0.74	0.02	4.47	3.83	0.11	22.5	11.96	45.8	2.4	548	2263
7	5.11	0.92	0.87	0.01	4.69	4.45	0.06	17.5	13.16	58.5	1.1	771	1436
8	5.08	0.95	0.93	0	4.81	4.69	0.02	12.5	14.06	65.9	0.2	927	752
9	5.03	0.98	0.97	0	4.92	4.88	0	7.5	14.66	71.5	0	1049	277
10	5.02	1.00	1.00	0	5.02	5.02	0	2.5	14.96	75.1	0	1124	31
					42.19	35.02				405	75.4	5161	29733
					x2=	x2=				x2=	x2=	x2=	x2=
					84.38	70.04				810	150.8	10322	59466
										659.2			
												6215	0
												4107	59466
													62086

$\bar{y} = \frac{659.2}{70.04} = 9.42$ ft.

Corrections to Centroid

$v_{zi} = \frac{9213}{62086} = 0.148$ kips

At A

$m_{xi} = -\frac{75.2}{70.04} + 0.148(-9.42) = -1.07 - 1.39 = -2.46$ ft.-kips

$m_{yi} = \frac{217.9}{84.38} - 0.148(-50) = 2.59 + 7.4 = 9.99$ ft.-kips

$v_z = 1 - 0.148 = 0.852$ kips

$m_x = -11.25 + 2.46 = -8.79$ ft.-kips

$m_y = 25 - 9.99 = 15.01$ ft.-kips

At B

$m_{xi} = -2.46$ ft.-kips

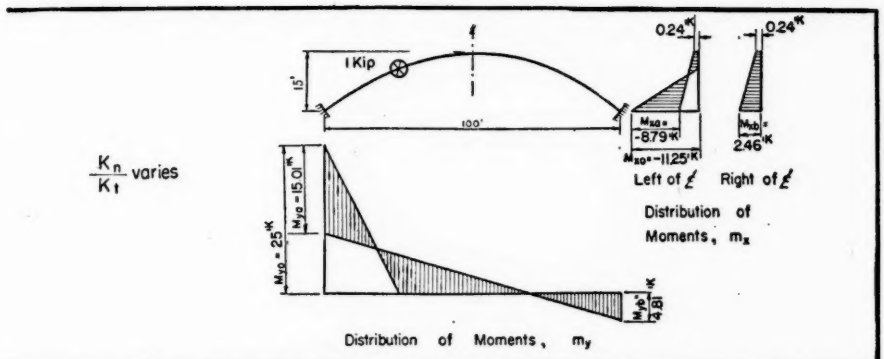
$m_{yi} = 2.59 - 7.4 = -4.81$ ft.-kips

$v_z = 0 - 0.148 = -0.148$ kips

$m_x = 0 - (-2.46) = 2.46$ ft.-kips

$m_y = 0 - (-4.81) = 4.81$ ft.-kips

FIG. 7.—PARABOLIC ARCH OF VARYING



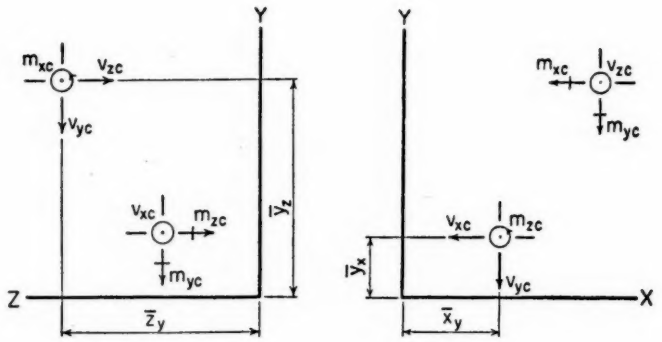
$xyda_{xy}$	m_{xo} (—)	m_{yo}	$m_{xo}da_{xx}$ (—)	$m_{yo}da_{xy}$ (—)	$m_{yo}da_{yy}$	$m_{xo}da_{xy}$	$ym_{xo}da_{xx}$ (—)	$ym_{yo}da_{xy}$ (—)	$xm_{yo}da_{yy}$ (—)	$xm_{xo}da_{xy}$ (—)
37.5	9.79	22.5	12.5	12.2	58.1	5.3	18.3	17.8	2760	251
87.5	7.09	17.5	12.8	8.7	58.1	3.5	53.0	36.0	2470	149
81.0	4.69	12.5	12.4	4.1	48.6	1.5	81.2	27.0	1824	58
68.6	2.59	7.5	8.0	1.8	31.3	0.6	69.3	15.9	1019	21
50.1	0.69	2.5	2.3	0.4	10.8	0.1	24.1	4.6	297	3
28.6	0	0	0	0	0	0	0	0	0	0
14.1	0	0	0	0	0	0	0	0	0	0
3.4	0	0	0	0	0	0	0	0	0	0
0.5	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
371.3			-48.0	-27.2	206.9	11.0	-245.9	-101.3	-8370	-482
x2= 742.6										
			-75.2		217.9		-347.2		-8852	
0							-707.9		0	
743							361		-8852	

9213

Section	1	2	3	4	5	6	7	8	9	10
$\frac{K_n}{K_t}$	0.60	0.67	0.78	0.83	0.86	0.91	0.94	0.97	0.99	1.00
Rel. K_n	1.76	1.47	1.31	1.23	1.18	1.13	1.08	1.05	1.02	1.00

CROSS SECTION WITH UNSYMMETRICAL LOAD

tions and displacements along the centroidal axes are shown in Fig. 8(*b*). These rotations and displacements are the result of assumed moments and correction moments. The expressions are similar to those of Fig. 3 except that three dimensions are now involved, and additional stiffness factors are intro-



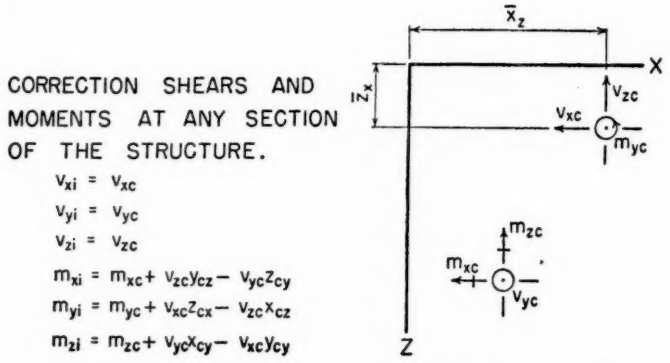
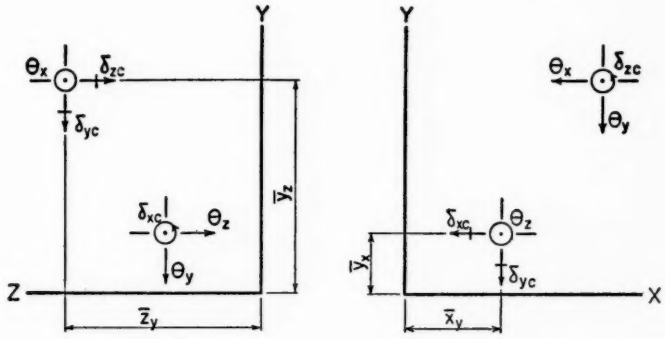


FIG. 8.—STATICAL AND GEOMETRICAL

duced. The stiffness factors for the general case of an element in space and oblique to the three orthogonal axes are summarized in Fig. 9.

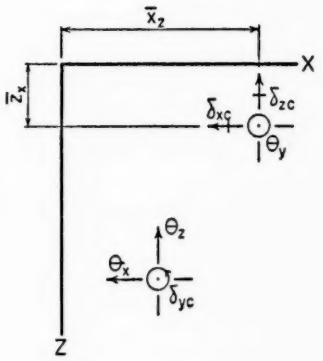
It follows from the requirements of geometry that the angle changes due to the correction moments must balance those caused by the assumed moments. From the relationships shown in Fig. 8, six equations of geometry may be writ-

ten. Each equation would involve the correction moments and the correction shears associated with the centroidal axes. It is convenient, however, to define the centroidal axes in such a way that a correction shear associated with one of these axes does not contribute to total rotations about axes normal to this



ROTATIONS AND DISPLACEMENTS DUE TO THE ASSUMED MOMENTS.

$$\theta_{xo} = \sum m_{xo}\, da_{xx} + \sum m_{yo} da_{xy} + \sum m_{zo} da_{xz}$$
$$\theta_{yo} = \sum m_{xo}\, da_{yx} + \sum m_{yo} da_{yy} + \sum m_{zo} da_{yz}$$
$$\theta_{zo} = \sum m_{xo}\, da_{zx} + \sum m_{yo} da_{zy} + \sum m_{zo} da_{zz}$$
$$\delta_{xco} = \sum z_{cx} d\theta_{yo} - \sum y_{cx} d\theta_{zo}$$
$$\delta_{yco} = \sum x_{cy} d\theta_{zo} - \sum z_{cy} d\theta_{xo}$$
$$\delta_{zco} = \sum y_{cz} d\theta_{xo} - \sum x_{cz} d\theta_{yo}$$



ROTATIONS AND DISPLACEMENTS DUE TO THE CORRECTION MOMENTS.

$$\theta_{xi} = \sum m_{xi}\, da_{xx} + \sum m_{yi}\, da_{xy} + \sum m_{zi}\, da_{xz}$$
$$\theta_{yi} = \sum m_{xi}\, da_{yx} + \sum m_{yi}\, da_{yy} + \sum m_{zi}\, da_{yz}$$
$$\theta_{zi} = \sum m_{xi}\, da_{zx} + \sum m_{yi}\, da_{zy} + \sum m_{zi}\, da_{zz}$$
$$\delta_{xci} = \sum z_{cx} d\theta_{yi} - \sum y_{cx} d\theta_{zi}$$
$$\delta_{yci} = \sum x_{cy} d\theta_{zi} - \sum z_{cy} d\theta_{xi}$$
$$\delta_{zci} = \sum y_{cz} d\theta_{xi} - \sum x_{cz} d\theta_{yi}$$

RELATIONSHIPS FOR THE GENERAL PROCEDURE

shear. For example, a shear v_{zc} of one unit contributes no rotation about a y-axis or a z-axis if the distances $\bar{z}_x$ and $\bar{y}_x$ are selected such that

$$\sum d\theta_{yi} = \sum z_{cx}\, da_{yy} - \sum y_{cx}\, da_{yz} = 0 \quad \ldots\ldots\ldots\ldots (13a)$$

and

$$\sum d\theta_{zi} = \sum z_{cx}\, da_{zy} - \sum y_{cx}\, da_{zz} = 0 \quad \ldots\ldots\ldots\ldots (13b)$$

in which

$$z_{cx} = z - \bar{z}_x \quad \ldots\ldots (14a)$$

and

$$y_{cx} = y - \bar{y}_x \quad \ldots\ldots (14b)$$

The expressions for $\bar{z}_x$ and $\bar{y}_x$ may be obtained by Eqs. 13 and 14. The expressions for $\bar{x}_y$ and $\bar{z}_y$ associated with v_{yc}, and for $\bar{y}_z$ and $\bar{x}_z$ associated with v_{zc}, can be obtained from similar equations.

The six equations of geometry may now be written as follows:

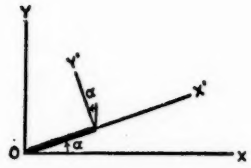
$$\left.\begin{aligned}
\textstyle\sum d\theta_x &= A_x\, m_{xc} + A_{xy}\, m_{yc} + A_{xz}\, m_{zc} + f_{xx}\, v_{xc} - P_x = 0 \\
\textstyle\sum d\theta_y &= A_{yx}\, m_{xc} + A_y\, m_{yc} + A_{yz}\, m_{zc} + f_{yy}\, v_{yc} - P_y = 0 \\
\textstyle\sum d\theta_z &= A_{zx}\, m_{xc} + A_{zy}\, m_{yc} + A_z\, m_{zc} + f_{zz}\, v_{zc} - P_z = 0 \\
\delta_x &= t_{xx}\, m_{xc} + J_x\, v_{xc} + J_{xy}\, v_{yc} + J_{xz}\, v_{zc} - M_x = 0 \\
\delta_y &= t_{yy}\, m_{yc} + J_{yx}\, v_{xc} + J_y\, v_{yc} + J_{yz}\, v_{zc} - M_y = 0 \\
\delta_z &= t_{zz}\, m_{zc} + J_{zx}\, v_{xc} + J_{zy}\, v_{yc} + J_z\, v_{zc} - M_z = 0
\end{aligned}\right\} \quad \ldots\ldots (15)$$

in which the coefficients of the correction moments and shears are summarized in Figs. 10 and 11. These coefficients may be interpreted as total rotations and displacements due to unit moments and shears associated with the centroidal axes. For example, A_{xy} represents a rotation about an x-axis due to a moment m_{yc} equal to one unit; the term f_{xx} represents a rotation about an x-axis due to a shear v_{xc} equal to one unit; and the term t_{xx} represents a displacement along the centroidal axis associated with v_{xc} and is due to a moment m_{xc} equal to one unit. It can be shown that $t_{xx} = f_{xx}$, $t_{yy} = f_{yy}$, and $t_{zz} = f_{zz}$. The term J_{xy} represents a displacement along the centroidal axis associated with v_{xc}. This displacement is caused by a shear v_{yc} equal to one unit. It can be shown that $A_{yx} = A_{xy}$ and $J_{yx} = J_{xy}$.

In the equations of geometry (Eqs. 15), the P-terms and M-terms represent rotations and displacements resulting from the assumed distributions of moments. For example, P_x represents the total rotation about an x-axis. The term M_x represents a displacement along the centroidal axis in the x-direction. These terms may also represent unit rotations and displacements applied to the structure as in the Müller-Breslau procedure for obtaining influence lines. In addition, the M-terms may represent displacements that would occur in a pipe line if it were free to expand when subjected to temperature changes.

The correction moments and the correction shear along the centroidal axes may now be obtained by solving the equations of geometry. Having developed the expressions for these correction moments and shears, the designer can obtain the correction moments and shears at any section of the structure by the equations of statics summarized in Fig. 8(a). The expressions for these shears and moments are written in the following form:

$$\left.\begin{aligned}
v_{xi} &= \frac{\mathfrak{M}'_x}{\mathfrak{J}'_x} \\
v_{yi} &= \frac{\mathfrak{M}'_y}{\mathfrak{J}'_y} \\
v_{zi} &= \frac{\mathfrak{M}'_z}{\mathfrak{J}'_z}
\end{aligned}\right\} \quad \ldots\ldots (16)$$



PROJECTION OF ELEMENT ds in XY PLANE

$$K_{xy} = \frac{K_{n1}K_{n2}K_t}{K_{n1}K_{n2}C_1 + K_{n1}K_t(C_2 + aC_3) - K_{n2}K_t(C_2 + bC_3)}$$

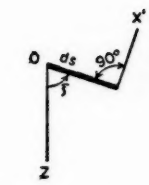
$$K_{yy} = \frac{K_{n1}K_{n2}K_t}{K_{n1}K_{n2}C_4 + K_{n1}K_t(C_5 + C_6) - K_{n2}K_t(C_5 - C_7)}$$

$$K_{yz} = \frac{K_{n1}K_{n2}K_t}{K_{n1}K_{n2}C_8 - K_{n1}K_t(C_9 + C_{10}) + K_{n2}K_t(C_9 - C_{11})}$$

$$K_{xx} = \frac{K_{n1}K_{n2}K_t}{K_{n1}K_{n2}C_{12} - K_{n1}K_t(C_5 - C_{13}) + K_{n2}K_t(C_5 + C_{14})}$$

$$K_{zx} = \frac{K_{n1}K_{n2}K_t}{K_{n1}K_{n2}C_{15} - K_{n1}K_t(C_{16} - C_{17}) - K_{n2}K_t(C_{17} + C_{18})}$$

$$K_{zz} = \frac{K_{n1}K_{n2}K_t}{K_{n1}K_{n2}C_{19} + K_{n1}K_tC_{20} + K_{n2}K_tC_{21}}$$



TRUE VIEW OF ELEMENT ds.

$C_1 = \sin\alpha\cos\alpha\sin^2\xi$

$C_2 = (\cos^2\alpha - \sin^2\alpha)\sin\psi\cos\psi\cos\xi$

$a = \cos^2\psi - \dfrac{\sin^2\psi}{\cos^2\xi}$

$C_3 = \sin\alpha\cos\alpha\cos^2\xi$

$b = \dfrac{\cos^2\psi}{\cos^2\xi} - \sin^2\psi$

$C_4 = \sin^2\alpha\sin^2\xi$

$C_5 = 2\sin\alpha\cos\alpha\sin\psi\cos\psi\cos\xi$

$C_6 = \sin^2\alpha\cos^2\psi\cos^2\xi + \cos^2\alpha\sin^2\psi$

$C_7 = \sin^2\alpha\sin^2\psi\cos^2\xi + \cos^2\alpha\cos^2\psi$

$C_8 = \sin\alpha\sin\xi\cos\xi$

$C_9 = \cos\alpha\sin\psi\cos\psi\sin\xi$

$C_{10} = \sin\alpha\cos^2\psi\sin\xi\cos\xi$

$C_{11} = \sin\alpha\sin^2\psi\sin\xi\cos\xi$

$C_{12} = \cos^2\alpha\sin^2\xi$

$C_{13} = \cos^2\alpha\cos^2\psi\cos^2\xi + \sin^2\alpha\sin^2\psi$

$C_{14} = \cos^2\alpha\sin^2\psi\cos^2\xi + \sin^2\alpha\cos^2\psi$

$C_{15} = \cos\alpha\sin\xi\cos\xi$

$C_{16} = \cos\alpha\cos^2\psi\sin\xi\cos\xi$

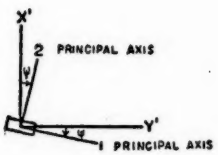
$C_{17} = \sin\alpha\sin\psi\cos\psi\sin\xi$

$C_{18} = \cos\alpha\sin^2\psi\sin\xi\cos\xi$

$C_{19} = \cos^2\xi$

$C_{20} = \cos^2\psi\sin^2\xi$

$C_{21} = \sin^2\psi\sin^2\xi$



TRUE END VIEW OF ELEMENT ds.

FOR THE SPECIAL CASE OF $\psi = 0$: A PRINCIPAL AXIS OF THE CROSS SECTION IS PARALLEL TO ONE OF THE REFERENCE PLANES.

$C_2 = 0$

$a = 1$

$b = 1/\cos^2\xi$

$C_5 = 0$

$C_6 = \sin^2\alpha\cos^2\xi$

$C_7 = \cos^2\alpha$

$C_9 = 0$

$C_{10} = \sin\alpha\sin\xi\cos\xi$

$C_{11} = 0$

$C_{13} = \cos^2\alpha\cos^2\xi$

$C_{14} = \sin^2\alpha$

$C_{16} = \cos\alpha\sin\xi\cos\xi$

$C_{17} = 0$

$C_{18} = 0$

$C_{20} = \sin^2\xi$

$C_{21} = 0$

FIG. 9.—STIFFNESS FACTORS FOR THE GENERAL CASE OF AN ELEMENT IN SPACE

Member	Length	$\frac{1}{K_{xx}}$	$\frac{1}{K_{yy}}$	$\frac{1}{K_{zz}}$	$\frac{1}{K_{xy}}$	$\frac{1}{K_{yz}}$	$\frac{1}{K_{zx}}$	da_{xx}	da_{yy}	da_{zz}	da_{xy}

Totals		A_x		A_y		A_z		A_{xy}
Corrections (subtract)		$\frac{A_{xy}^2}{A_y}$	$\frac{A_{zx}^2}{A_z}$	$\frac{A_{xy}^2}{A_x}$	$\frac{A_{yz}^2}{A_z}$	$\frac{A_{zx}^2}{A_x}$	$\frac{A_{yz}^2}{A_y}$	$\frac{A_{yz}A_{zx}}{A_z}$
		$(A'_x)_y$	$(A'_x)_z$	$(A'_y)_x$	$(A'_y)_z$	$(A'_z)_x$	$(A'_z)_y$	A'_{xy}

Centroids

$$\bar{z}_x = \frac{(C'_y)_z}{(A'_y)_z} \qquad \bar{x}_y = \frac{(C'_z)_x}{(A'_z)_x} \qquad \bar{y}_z = \frac{(C'_x)_y}{(A'_x)_y}$$

$$\bar{y}_x = -\frac{(C'_z)_y}{(A'_z)_y} \qquad \bar{z}_y = -\frac{(C'_x)_z}{(A'_x)_z} \qquad \bar{x}_z = -\frac{(C'_y)_x}{(A'_y)_x}$$

$z\,da_{xy}$		$x\,da_{yz}$		$y\,da_{zx}$		$(I_x)_z + y^2 da_{xx}$	$(I_y)_z + x^2 da_{yy}$	$(I_{xy})_z + xy\,da_{xy}$	$(I_y)_x + z^2 da_{yy}$	$(I_z)_x + y^2 da_{zz}$	$(I_{yz})_x + yz\,da_{yz}$	$(I_z)_y + x^2 da_{zz}$	$(I_x)_y + z^2 da_{xx}$	$(I_{zx})_y + zx\,da_{zx}$
Corrections (subtract):-														
$\bar{x}_x \cdot A_{xy}$	$\bar{z}_y \cdot A_{xy}$	$\bar{x}_y \cdot A_{yz}$	$\bar{x}_z \cdot A_{yz}$	$\bar{y}_z \cdot A_{zx}$	$\bar{y}_x \cdot A_{zx}$	$\bar{y}_z^2 A_x$	$\bar{x}_z^2 A_y$	$\bar{x}_z \bar{y}_z A_{xy}$	$\bar{z}_x^2 A_y$	$\bar{y}_x^2 A_z$	$\bar{y}_x \bar{z}_x A_{yz}$	$\bar{x}_y^2 A_z$	$\bar{z}_y^2 A_x$	$\bar{z}_y \bar{x}_y A_{zx}$
①	②	③	④	⑤	⑥	$(I_x)_z$	$(I_y)_z$	$(I_{xy})_z$	$(I_y)_x$	$(I_z)_x$	$(I_{yz})_x$	$(I_z)_y$	$(I_x)_y$	$(I_{zx})_y$

①		③		⑤		$J_z = (I_x)_z + (I_y)_z - 2(I_{xy})_z$		$J_x = (I_y)_x + (I_z)_x - 2(I_{yz})_x$		$J_y = (I_z)_y + (I_x)_y - 2(I_{zx})_y$	
Subtract :-											
⑥		②		④		$\frac{J_{zx}^2}{J_x}$	$\frac{J_{yz}^2}{J_y}$	$\frac{J_{xy}^2}{J_y}$	$\frac{J_{zx}^2}{J_z}$	$\frac{J_{yz}^2}{J_z}$	$\frac{J_{xy}^2}{J_x}$
t_{xx}		t_{yy}		t_{zz}		$(J'_z)_x$	$(J'_z)_y$	$(J'_x)_y$	$(J'_x)_z$	$(J'_y)_z$	$(J'_y)_x$

$$J'_x = J_x - J_{xy}\frac{J'_{xy}}{(J'_y)_z} - J_{zx}\frac{J'_{zx}}{(J'_z)_y} \qquad A_x = A_x - \frac{t_{xx}^2}{J'_x} \qquad A_{xy} = A_{xy} + \frac{t_{xx}t_{yy}J'_{xy}}{J'_x(J'_y)_z}$$

$$J'_y = J_y - J_{yz}\frac{J'_{yz}}{(J'_z)_x} - J_{xy}\frac{J'_{xy}}{(J'_x)_z} \qquad A_y = A_y - \frac{t_{yy}^2}{J'_y} \qquad A_{yz} = A_{yz} + \frac{t_{yy}t_{zz}J'_{yz}}{J'_y(J'_z)_x}$$

$$J'_z = J_z - J_{zx}\frac{J'_{zx}}{(J'_x)_y} - J_{yz}\frac{J'_{yz}}{(J'_y)_x} \qquad A_z = A_z - \frac{t_{zz}^2}{J'_z} \qquad A_{zx} = A_{zx} + \frac{t_{zz}t_{xx}J'_{zx}}{J'_z(J'_x)_y}$$

FIG. 10.—GENERAL PROCEDURE FOR ANALYSIS

da_{yz}	da_{zx}	x	y	z	$z da_{yy}$	$y da_{yz}$	$z da_{yz}$	$y da_{zz}$	$x da_{zz}$	$z da_{zx}$	$x da_{zx}$	$z da_{xx}$	$y da_{xx}$	$x da_{xy}$	$y da_{xy}$	$x da_{yy}$
					z	z'	z	z'	z	z'	z	z'	z	z'	z	z'
A_{yz}	A_{zx}				$z-z'=(C_y)_z$		$z-z'=(C_z)_y$		$z-z'=(C_z)_x$		$z-z'=(C_z)_z$		$z-z'=(C_x)_y$		$z-z'=(C_y)_x$	
$\frac{A_{zx}A_{xy}}{A_x}$	$\frac{A_{xy}A_{yz}}{A_y}$				$\frac{A_{yz}}{A_z}(C_z)_y$		$\frac{A_{yz}}{A_y}(C_y)_z$		$\frac{A_{zx}}{A_x}(C_z)_z$		$\frac{A_{zx}}{A_z}(C_z)_x$		$\frac{A_{xy}}{A_y}(C_y)_x$		$\frac{A_{xy}}{A_x}(C_x)_y$	
A'_{yz}	A'_{zx}				$(C'_y)_z$		$(C'_z)_y$		$(C'_z)_x$		$(C'_x)_z$		$(C'_x)_y$		$(C'_y)_x$	

$$A'_x = A_x - A_{xy}\frac{A'_{xy}}{(A'_y)_z} - A_{zx}\frac{A'_{zx}}{(A'_z)_y} \qquad A'_y = A_y - A_{yz}\frac{A'_{yz}}{(A'_z)_x} - A_{xy}\frac{A'_{xy}}{(A'_x)_z} \qquad A'_z = A_z - A_{zx}\frac{A'_{zx}}{(A'_x)_y} - A_{yz}\frac{A'_{yz}}{(A'_y)_x}$$

$(i_{xy})_{xy}$ $+z^2 da_{xy}$	$(i_{yz})_{xy}$ $+zx\,da_{yz}$	$(i_{zx})_{xy}$ $+yz\,da_{zx}$	$(i_z)_{xy}$ $+xy\,da_{zz}$	$(i_{yz})_{yz}$ $+x^2 da_{yz}$	$(i_{zx})_{yz}$ $+xy\,da_{zx}$	$(i_{xy})_{yz}$ $+zx\,da_{xy}$	$(i_x)_{yz}$ $+yz\,da_{xx}$	$(i_{zx})_{zx}$ $+y^2 da_{zx}$	$(i_{xy})_{zx}$ $+yz\,da_{xy}$	$(i_{yz})_{zx}$ $+xy\,da_{yz}$	$(i_y)_{zx}$ $+zx\,da_{yy}$
z	z	z	z	z	z	z	z	z	z	z	z
$\bar{z}_{xy}\sum z\,da_{xy}$ $+\bar{z}_y\sum z\,da_{xy}$ $-\bar{z}_x\bar{z}_y\sum da_{xy}$	$\bar{z}_x\sum x\,da_{yz}$	$\bar{z}_y\sum y\,da_{zx}$	$\bar{x}_y\bar{y}_x\sum da_{zz}$	$\bar{x}_y\sum x\,da_{yz}$ $+\bar{x}_z\sum x\,da_{yz}$ $-\bar{x}_y\bar{x}_z\sum da_{yz}$	$\bar{x}_y\sum y\,da_{zx}$	$\bar{x}_z\sum z\,da_{xy}$	$\bar{y}_z\bar{z}_y\sum da_{xx}$	$\bar{y}_z\sum y\,da_{zx}$ $+\bar{y}_z\sum y\,da_{zx}$ $-\bar{y}_z\bar{y}_x\sum da_{zx}$	$\bar{y}_z\sum z\,da_{xy}$	$\bar{y}_x\sum x\,da_{yz}$	$\bar{z}_x\bar{z}_y\sum da_{yy}$
$(I_{xy})_{xy}$	$(I_{yz})_{xy}$	$(I_{zx})_{xy}$	$(I_z)_{xy}$	$(I_{yz})_{yz}$	$(I_{zx})_{yz}$	$(I_{xy})_{yz}$	$(I_x)_{yz}$	$(I_{zx})_{zx}$	$(I_{xy})_{zx}$	$(I_{yz})_{zx}$	$(I_y)_{zx}$
$J_{xy} = -(I_{xy})_{xy} + (I_{yz})_{xy} + (I_{zx})_{xy} - (I_z)_{xy}$ $\frac{J_{yz}J_{zx}}{J_z}$				$J_{yz} = -(I_{yz})_{yz} + (I_{zx})_{yz} + (I_{xy})_{yz} - (I_x)_{yz}$ $\frac{J_{zx}J_{xy}}{J_x}$				$J_{zx} = -(I_{zx})_{zx} + (I_{xy})_{zx} + (I_{yz})_{zx} - (I_y)_{zx}$ $\frac{J_{xy}J_{yz}}{J_y}$			
J'_{xy}				J'_{yz}				J'_{zx}			

$$\mathfrak{J}_x = J_x - \frac{I^2_{xx}}{A'_x} \qquad \mathfrak{J}_{xy} = J_{xy} + \frac{I_{xx}I_{yy}A'_{xy}}{A'_x(A'_y)_z}$$

$$\mathfrak{J}_y = J_y - \frac{I^2_{yy}}{A'_y} \qquad \mathfrak{J}_{yz} = J_{yz} + \frac{I_{yy}I_{zz}A'_{yz}}{A'_y(A'_z)_x}$$

$$\mathfrak{J}_z = J_z - \frac{I^2_{zz}}{A'_z} \qquad \mathfrak{J}_{zx} = J_{zx} + \frac{I_{zz}I_{xx}A'_{zx}}{A'_z(A'_x)_y}$$

$$A'_{xx} = A_{xx} - \frac{A_{xy}A_{yx}}{A_y} \qquad (A'_x)_y = A_x - \frac{A_{xy}^2}{A_y} \qquad (A'_x)_z = A_x - \frac{A_{zx}^2}{A_z} \qquad A'_x = A_x - A_{xy}\frac{A'_{xy}}{(A'_y)_x} - A_{xz}\frac{A'_{zx}}{(A'_z)_y}$$

$$A'_{xy} = A_{xy} - \frac{A_{xz}A_{zy}}{A_z} \qquad (A'_x)_z = A_x - \frac{A_{xz}^2}{A_z} \qquad (A'_y)_x = A_y - \frac{A_{yx}^2}{A_x} \qquad A'_y = A_y - A_{yz}\frac{A'_{yz}}{(A'_z)_x} - A_{xy}\frac{A'_{xy}}{(A'_x)_z}$$

$$A'_{yz} = A_{yz} - \frac{A_{zx}A_{xy}}{A_x} \qquad (A'_y)_z = A_y - \frac{A_{xy}^2}{A_x} \qquad (A'_y)_x = A_y - \frac{A_{yz}^2}{A_z} \qquad A'_z = A_z - A_{zx}\frac{A'_{zx}}{(A'_x)_y} - A_{yz}\frac{A'_{yz}}{(A'_y)_x}$$

$m_{xo}da_{xx}$	$m_{yo}da_{xy}$	$m_{zo}da_{zx}$	$m_{xo}da_{xy}$	$m_{yo}da_{yy}$	$m_{zo}da_{yz}$	$m_{xo}da_{zx}$	$m_{yo}da_{yz}$	$m_{zo}da_{zz}$	$zm_{xo}da_{xy}$	$zm_{yo}da_{yy}$	$zm_{zo}da_{yz}$	$ym_{xo}da_{zx}$	$ym_{yo}da_{yz}$
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
$\downarrow+\downarrow+\downarrow = P_x$			$\downarrow+\downarrow+\downarrow = P_y$			$\downarrow+\downarrow+\downarrow = P_z$			$\downarrow+\downarrow-\downarrow$			$\downarrow+\downarrow-\downarrow$	
Subtract :-													
$P_y \frac{A'_{xy}}{(A'_y)_z} + P_z \frac{A'_{zx}}{(A'_z)_y}$			$P_z \frac{A'_{yz}}{(A'_z)_x} + P_x \frac{A'_{xy}}{(A'_x)_z}$			$P_x \frac{A'_{zx}}{(A'_x)_y} + P_y \frac{A'_{yz}}{(A'_y)_x}$			$\bar{z}_x P_y$			$\bar{y}_x P_z$	
									↓			↓	↓
P'_x			P'_y			P'_z			$\downarrow-\downarrow-\downarrow = M_x$				
									Subtract : $M_y \frac{J'_{xy}}{(J'_y)_z} + M_z \frac{J'_{zx}}{(J'_z)_y}$				
									M'_x				

$$F_x = t_{xx}\frac{M'_x}{J'_x} \qquad T_x = t_{xx}\frac{P'_x}{A'_x}$$

$$F_y = t_{yy}\frac{M'_y}{J'_y} \qquad T_y = t_{yy}\frac{P'_y}{A'_y}$$

$$F_z = t_{zz}\frac{M'_z}{J'_z} \qquad T_z = t_{zz}\frac{P'_z}{A'_z}$$

$$F_x = F_x - F_y\frac{A_{xy}'}{(A_y')_z} - F_z\frac{A_{zx}'}{(A_z')_y} \qquad T_x = T_x - T_y\frac{J_{xy}'}{(J_y')_z} - T_z\frac{J_{zx}'}{(J_z')_y} \qquad P_x = P_x - P_y\frac{A_{xy}'}{(A_y')_z} - P_z\frac{A_{zx}'}{(A_z')_y}$$

$$F_y = F_y - F_z\frac{A_{yz}'}{(A_z')_x} - F_x\frac{A_{xy}'}{(A_x')_z} \qquad T_y = T_y - T_z\frac{J_{yz}'}{(J_z')_x} - T_x\frac{J_{xy}'}{(J_x')_z} \qquad P_y = P_y - P_z\frac{A_{yz}'}{(A_z')_x} - P_x\frac{A_{xy}'}{(A_x')_z}$$

$$F_z = F_z - F_x\frac{A_{zx}'}{(A_x')_y} - F_y\frac{A_{yz}'}{(A_y')_x} \qquad T_z = T_z - T_x\frac{J_{zx}'}{(J_x')_y} - T_y\frac{J_{yz}'}{(J_y')_x} \qquad P_z = P_z - P_x\frac{A_{zx}'}{(A_x')_y} - P_y\frac{A_{yz}'}{(A_y')_x}$$

$$v_{xi} = \frac{M'_x}{J'_x} \qquad m_{xi} = \frac{P'_x}{A'_x} + \frac{M'_z}{J'_z}y_{cz} - \frac{M'_y}{J'_y}z_{cy} \qquad v_x = v_{xo} - v_{xi} \qquad m_x = m_{xo} - m_{xi}$$

$$v_{yi} = \frac{M'_y}{J'_y} \qquad m_{yi} = \frac{P'_y}{A'_y} + \frac{M'_x}{J'_x}z_{cx} - \frac{M'_z}{J'_z}x_{cz} \qquad v_y = v_{yo} - v_{yi} \qquad m_y = m_{yo} - m_{yi}$$

$$v_{zi} = \frac{M'_z}{J'_z} \qquad m_{zi} = \frac{P'_z}{A'_z} + \frac{M'_y}{J'_y}x_{cy} - \frac{M'_x}{J'_x}y_{cx} \qquad v_z = v_{zo} - v_{zi} \qquad m_z = m_{zo} - m_{zi}$$

FIG. 11.—GENERAL PROCEDURE FOR ANALYSIS

$$J'_{zx} = J_{zx} - \frac{J_{xy}J_{yz}}{J_y} \qquad (J'_x)_y = J_x - \frac{J_{xy}^2}{J_y} \qquad (J'_x)_z = J_x - \frac{J_{zx}^2}{J_z} \qquad J'_x = J_x - J_{xy}\frac{J'_{xy}}{(J'_y)_x} - J_{zx}\frac{J'_{zx}}{(J'_z)_y}$$

$$J'_{xy} = J_{xy} - \frac{J_{yz}J_{zx}}{J_z} \qquad (J'_x)_z = J_x - \frac{J_{xy}^2}{J_z} \qquad (J'_z)_y = J_z - \frac{J_{yz}^2}{J_y} \qquad J'_y = J_y - J_{xy}\frac{J'_{yz}}{(J'_z)_x} - J_{xy}\frac{J'_{xy}}{(J'_z)_z}$$

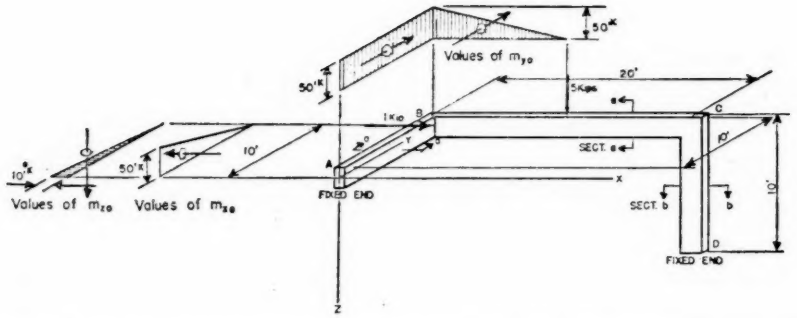
$$J'_{yz} = J_{yz} - \frac{J_{zx}J_{xy}}{J_x} \qquad (J'_y)_x = J_y - \frac{J_{xy}^2}{J_x} \qquad (J'_y)_z = J_y - \frac{J_{yz}^2}{J_z} \qquad J'_z = J_z - J_{zx}\frac{J'_{zx}}{(J'_x)_y} - J_{yz}\frac{J'_{yz}}{(J'_y)_x}$$

$ym_z da_{zz}$	$xm_x da_{zx}$	$xm_y da_{yz}$	$xm_z da_{zz}$	$zm_x da_{xx}$	$zm_y da_{xy}$	$zm_z da_{zx}$	$ym_x da_{xx}$	$ym_y da_{xy}$	$ym_z da_{zx}$	$xm_x da_{zy}$	$xm_y da_{yy}$	$xm_z da_{yz}$
d	d	d	d	d	d	d	d	d	d	d	d	d
	$d+d+d$			$d+d+d$			$d+d+d$			$d+d+d$		
	$\bar{x}_y P_z$			$\bar{z}_y P_x$			$\bar{y}_z P_x$			$\bar{x}_z P_y$		
	d			$d\,d$			d			$d\,d$		
	$d-dd=M_y$						$d-N=M_z$					
	$M_z\frac{J'_{yz}}{(J'_z)_x} + M_x\frac{J'_{xy}}{(J'_x)_z}$						$M_x\frac{J'_{zx}}{(J'_x)_y} + M_y\frac{J'_{yz}}{(J'_y)_x}$					
	M'_y						M'_z					

$$M_x = M_x - M_y\frac{J'_{xy}}{(J'_y)_z} - M_z\frac{J'_{zx}}{(J'_z)_y} \qquad P'_x = P_x - F_x \qquad M'_x = M_x - T_x$$

$$M_y = M_y - M_z\frac{J'_{yz}}{(J'_z)_x} - M_x\frac{J'_{xy}}{(J'_x)_z} \qquad P'_y = P_y - F_y \qquad M'_y = M_y - T_y$$

$$M_z = M_z - M_x\frac{J'_{zx}}{(J'_x)_y} - M_y\frac{J'_{yz}}{(J'_y)_x} \qquad P'_z = P_z - F_z \qquad M'_z = M_z - T_z$$



MEMBER	LENGTH FT	$1/K_{xx}$ $\times10^9$	$1/K_{yy}$ $\times10^9$	$1/K_{zz}$ $\times10^9$	$d\alpha_{xx}$ $\times10^9$	$d\alpha_{yy}$ $\times10^9$	$d\alpha_{zz}$ $\times10^9$	x FT	y FT	z FT	$z\,d\alpha_{yy}$ $\times10^9$	$y\,d\alpha_{zz}$ $\times10^9$	$x\,d\alpha_{zz}$ $\times10^9$	$z\,d\alpha_{xx}$ $\times10^9$	$y\,d\alpha_{xx}$ $\times10^9$	$x\,d\alpha_{yy}$ $\times10^9$	$(i_x)_z + y^2 d\alpha_{xx}$ $\times10^7$	$(i_y)_z + x^2 d\alpha_{yy}$ $\times10^7$
AB	10	3.33	10.00	10.00	33.3	100.0	100.0	0	5	0	0	500	0	0	167	0	11.1	0
BC	20	10.00	3.33	10.00	200.0	66.6	200.0	10	10	0	0	2000	2000	0	2000	666	200.0	88.8
CD	10	10.00	3.33	10.00	100.0	33.3	100.0	20	10	5	166.5	1000	2000	500	1000	666	100.0	133.2
				TOTALS	A_x= 333.3	A_y= 200.0	A_z= 400.0				166.5	3500	4000	500	3167	1332	311.1	222.0
										TOTALS	$(C_y)_z$= 166.5	$(C_z)_y$= -3500	$(C_z)_x$= 4000	$(C_x)_z$= -500	$(C_x)_y$= 3167	$(C_y)_x$= -1332		
																SUBTRACT	300.5	88.7
																	$(I_x)_z$= 10.6	$(I_y)_z$= 133.3
																	J_z = 143.9	
																SUBTRACT	13.5	0.02
																	$(J_z')_x$= 130.4	$(J_z')_y$= 143.9

$\bar{X}_x = \frac{166.5}{200} = 0.833$ ft

$\bar{Y}_x = \frac{-3500}{400} = 8.75$ ft

$\bar{Y}_z = \frac{3167}{333} = 9.50$ ft

$\bar{X}_z = \frac{-1332}{200} = 6.66$ ft

$\bar{Y}_y = \frac{4000}{400} = 10.00$ ft

$\bar{Y}_y = \frac{-500}{333} = 1.50$ ft

$v_{xi} = \frac{M'_z}{J'_z} = \frac{32.6}{24.54} = 1.328$ Kip

$v_{yi} = \frac{M'_y}{J'_y} = \frac{8.8}{215.9} = 0.04$ Kip

$v_{zi} = \frac{M'_x}{J'_z} = \frac{396.3}{125.6} = 3.07$ Kip

$V_x = V_{xo} - V_{xi}$

$V_{xB} = 1 - 1.328 = -0.328$ Kip

$V_{xC} = 0 - 1.328 = -1.328$ Kip

$V_y = V_{yo} - V_{yi}$

$V_{yB} = 0 - 0.04 = -0.04$ Kip

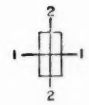
$V_{yC} = 0 - 0.04 = -0.04$ Kip

$V_z = V_{zo} - V_{zi}$

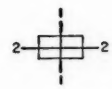
$V_{zB} = 5.00 - 3.07 = 1.93$ Kip

$V_{zC} = 0 - 3.07 = -3.07$ Kip

FIG. 12.—SPACE STRUCTURE



SECTION a-a



SECTION b-b

MEMBER AB

$K_{yy} = K_y = 1 \times 10^9$ lb ft²

$K_{xx} = K_{m1} = 3 \times 10^9$ lb ft²

$K_{zz} = K_{m2} = 1 \times 10^9$ lb ft²

MEMBER BC

$K_{yy} = K_{m1} = 3 \times 10^9$ lb ft²

$K_{xx} = K_y = 1 \times 10^9$ lb ft²

$K_{zz} = K_{m2} = 1 \times 10^9$ lb ft²

MEMBER CD

$K_{yy} = K_{m1} = 3 \times 10^9$ lb ft²

$K_{xx} = K_{m2} = 1 \times 10^9$ lb ft²

$K_{zz} = K_y = 1 \times 10^9$ lb ft²

$K_{m_1} = 3 \times 10^9$ lb ft²

$K_{m_2} = 1 \times 10^9$ lb ft²

$K_y = 1 \times 10^9$ lb ft²

$(I_y)_x + z^2 da_{yy}$ x 10^7	$(I_z)_x + y^2 da_{zz}$ x 10^7	$(I_z)_y + x^2 da_{zz}$ x 10^7	$(I_x)_y + z^2 da_{xx}$ x 10^7	$(I_z)_{xy} + xy da_{zz}$ x 10^7	$(I_x)_{yz} + yz da_{xx}$ x 10^7	$(I_y)_{zx} + zx da_{yy}$ x 10^7	$m_{xo} da_{xx}$ x 10^7	$m_{yo} da_{yy}$ x 10^7	$m_{zo} da_{zz}$ x 10^7	$z_i m_{yo} da_{yy}$ x 10^7	$y_i m_{zo} da_{zz}$ x 10^7	$x_i m_{zo} da_{zz}$ x 10^7	$z_i m_{xo} da_{xx}$ x 10^7	$y_i m_{zo} da_{xx}$ x 10^7	$x_i m_{yo} da_{yy}$ x 10^7
0	33.3	0	0	0	0	0	-8.33	50.00	5	0	16.6	0	0	-27.7	0
0	200.0	266.6	0	200	0	0	0	833	0	0	0	0	0	0	27.7
11.10	100.0	400.0	33.3	200	50	33.3	0	0	0	0	0	0	0	0	0
11.10	333.3	666.6	33.3	400	50	33.3	P_x = -8.33	P_y = 58.33	P_z = 5	0	16.6	0	0	-27.7	27.7
1.39	306.4	400.0	7.5	350	47.5	11.1		SUBTRACT		48.6	43.8	50	-12.5	-79.1	388.0
$(I_y)_x$ = 9.71	$(I_z)_x$ = 26.9	$(I_z)_y$ = 266.6	$(I_x)_y$ = 25.8	$(I_z)_{xy}$ = 50	$(I_x)_{yz}$ = 2.5	$(I_y)_{zx}$ = 22.2				-48.6	-27.2	-50	12.5	51.4	-360.3
J_x = 36.6		J_y = 292.4		J_{xy} = -50	J_{yz} = -2.5	J_{zx} = -22.2				M_x = -21.4		M_y = -62.5		M_z = 411.7	
8.55	3.4	0.04	68.3	0.39	30.3	0.43		SUBTRACT		10.8 + (-64.8) = -54.0		-103.8 + 32.5 = -71.3		17.25 + 9.16 = 26.41	
$(J_x')_y$ = 28.05	$(J_x')_z$ = 33.2	$(J_y')_z$ = 292.4	$(J_y')_x$ = 224.1	J_{xy}' = -50.4	J_{yz}' = -32.8	J_{zx}' = -22.6				M_x' = 32.6		M_y' = 8.8		M_z' = 385.3	

$$J_x' = 36.6 - (-50)\frac{-50.4}{292.4} - (-22.2)\frac{-22.6}{143.9} = 24.54$$

$$J_y' = 292.4 - (-25)\frac{-32.8}{130.4} - (-50)\frac{-50.4}{33.2} = 215.9$$

$$J_z' = 143.9 - (-22.2)\frac{-22.6}{26.05} - (-2.5)\frac{-32.8}{224.1} = 125.6$$

$$m_{xi} = \frac{P_x}{A_x} + \frac{M_z'}{J_z'} y_{cz} - \frac{M_y'}{J_y'} z_{cy}$$

$$m_{xo} = \frac{-8.33}{3.33} + 3.07(-9.5) - 0.04(-1.3) = -31.60'K$$

$$m_{xd} = \frac{-8.33}{3.33} + 3.07(0.5) - 0.04(8.5) = -1.30'K$$

$$m_x = m_{xo} - m_{xi}$$

$$m_{xo} = -50 - (-31.60) = -18.40'K$$

$$m_{xd} = 0 - (-1.30) = 1.30'K$$

$$m_{yi} = \frac{P_y}{A_y} + \frac{M_x'}{J_x'} z_{cx} - \frac{M_z'}{J_z'} z_{cz}$$

$$m_{yio} = \frac{58.33}{2} + 1.328(-0.833) - 3.07(-6.66) = 48.50'K$$

$$m_{yid} = \frac{58.33}{2} + 1.328(9.167) - 3.07(13.34) = 0.39'K$$

$$m_y = m_{yo} - m_{yi}$$

$$m_{yo} = 50 - 48.50 = 1.50'K$$

$$m_{yd} = 0 - 0.39 = -0.39'K$$

$$m_{zi} = \frac{P_z}{A_z} + \frac{M_y'}{J_y'} x_{cy} - \frac{M_x'}{J_x'} y_{cx}$$

$$m_{zio} = \frac{5}{4} + 0.04(-10) - 1.328(-8.75) = 12.48'K$$

$$m_{zid} = \frac{-5}{4} + 0.04(10) - 1.328(1.25) = -0.01'K$$

$$m_z = m_{zo} - m_{zi}$$

$$m_{zo} = 10 - 12.48 = -2.48'K$$

$$m_{zd} = 0 - (-0.01) = 0.01'K$$

WITH CONCENTRATED LOADS

$$\left.\begin{aligned} m_{xi} &= \frac{\mathcal{P}'_x}{\mathcal{A}'_x} + \frac{\mathfrak{M}'_z}{\mathcal{J}'_z} y_{cz} - \frac{\mathfrak{M}'_y}{\mathcal{J}'_y} z_{cy} \\ m_{yi} &= \frac{\mathcal{P}'_y}{\mathcal{A}'_y} + \frac{\mathfrak{M}'_x}{\mathcal{J}'_y} z_{cx} - \frac{\mathfrak{M}'_z}{\mathcal{J}'_z} x_{cz} \\ m_{zi} &= \frac{\mathcal{P}'_z}{\mathcal{A}'_z} + \frac{\mathfrak{M}'_y}{\mathcal{J}'_y} x_{cy} - \frac{\mathfrak{M}'_x}{\mathcal{J}'_x} y_{cx} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

The form of Eqs. 16 and 17 resulted from a consideration of the concept that permits the interchange of a rotation with an analogous force, and a displacement with an analogous moment. In these expressions, the $\mathcal{P}'$-terms represent rotations resulting from the assumed distributions of moments and may be considered to represent analogous forces. The $\mathfrak{M}'$-terms represent displacements along centroidal axes due to the assumed distributions of moments and may be considered to represent analogous moments. In the same manner as before, the $\mathcal{A}'$-terms and the $\mathcal{J}'$-terms represent rotations and displacements caused by moments and shears applied along the centroidal axes. The expressions for these rotations and displacements resemble expressions for areas and inertias.

Figs. 10 and 11 indicate, in tabular form, a convenient sequence for computing moments and shears in a structure curved in space and continuous between two supports. This form also serves as a summary of all terms involved in the analytical procedure. Inspection of these figures reveals that, in the special case of a structure lying in a plane, the expressions for the correction moments and shears become those of the column analogy or those of the shear and torsion analogy. Inspection also reveals that all script terms resulted from introducing the t-terms and f-terms in Eqs. 15. In numerical studies, the t-terms and f-terms are frequently negligible. In such cases, the script terms need not be computed and, consequently, Eqs. 18 and 19 may be used in computing the correction shears and moments:

$$\left.\begin{aligned} v_{xi} &= \frac{M'_x}{J'_x} \\ v_{yi} &= \frac{M'_y}{J'_y} \\ v_{zi} &= \frac{M'_z}{J'_z} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

$$\left.\begin{aligned} m_{xi} &= \frac{P'_x}{A'_x} + \frac{M'_z}{J'_z} y_{cz} - \frac{M'_y}{J'_y} z_{cy} \\ m_{yi} &= \frac{P'_y}{A'_y} + \frac{M'_x}{J'_x} z_{cx} - \frac{M'_z}{J'_z} x_{cz} \\ m_{zi} &= \frac{P'_z}{A'_z} + \frac{M'_y}{J'_y} x_{cy} - \frac{M'_x}{J'_x} y_{cx} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

Numerical Example.—Fig. 12 shows a structure consisting of three prismatic members, each of which is parallel to a rectangular axis. This structure is fixed at ends A and D, and is subjected to two concentrated loads. The assumed distributions of moments are for a structure cantilevered from end A.

The appropriate properties and the analogous forces and moments are tabulated as shown in Figs. 10 and 11. Since the members are parallel to the rectangular axes, many operations indicated in the latter figures need not be performed. For example, all operations involving terms written in script form may be omitted. In computing for segments of finite length the value of such a term as $(I_x)_z$, it may be necessary to add a correction $(i_x)_z$ to the value of $y^2\,da_{xx}$. This correction becomes necessary if the distance to each differential element of a finite segment is not the same as the tabulated y-distance to the centroid of the segment. Corrections such as these may be computed as moments and products of inertia of the elastic areas about their own centroidal axes. Computations for the final moments and shears at fixed ends A and D are shown at the bottom of Fig. 12. Final curves of moments may be drawn by combining the assumed distributions of moments with the planar distributions of correction moments.

If the effects of temperature changes are of interest, moments and shears in the structure may be computed in the same way as indicated in Fig. 12. The computations for the properties of the structure are exactly the same as before. The analogous moments M_x, M_y, and M_z are again equal to displacements in the x-direction, the y-direction, and the z-direction, respectively. However, these terms now represent the displacements of end D with respect to end A which would occur if the structure was free to expand. In this case, the analogous forces P_x, P_y, and P_z are equal to zero. The succeeding computations for moments and shears in the structure are identical in sequence to those indicated in Fig. 12. If the segments are not parallel to rectangular axes, the general form shown in Figs. 10 and 11 should be used.

Summary

It has been shown that a shear and torsion analogy may be used in analyzing any plane structure continuous between two supports and loaded normal to its plane. The results may be interpreted in terms of a pressure line concept—that is, the balance of angle changes in a closed circuit. This balance may be conveniently checked by substituting computations of statics for those of geometry.

It has also been shown that structures curved or segmental in space and continuous between two supports may be analyzed by the methods presented under the heading, "A General Procedure for Structures Curved in Space." The shear and torsion analogy and the column analogy are special cases of this procedure.

Acknowledgment

The writers wish to acknowledge their indebtedness to Mr. Cross. It was he who suggested that, for structures lying in a plane and loaded normal to their plane, there is merit in drawing curves of moments along rectangular axes.

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